Islamic Azad University

Journal of
Optoelectronical Nanostructures

Winter ٢٠٢١ / Vol. ٦, No. ١

IRANIAN SOLAR ENERGY SCIENTIFIC SOCIETY

Research Paper (Pape Type)

Biexciton in Strongly Oblate Ellipsoidal Quantum Dot with Relativistic Corrections

Received:
Revised:
Accepted:
Published:

Use your device to scan and read the article online

DOI:

Keywords:
Exciton, Ellipsoidal QD, Ground State Energy, Oblate Ellipsoid, Relativistic Correction.

Abstract

Recent progress in high-technology equipment enables the fabrication of quantum dots such as GaAs, and GaAlAs confining a finite number of excitons and allowing for control of the properties of quantum dots. Biexciton quantum dots are the simplest example that can be used to upgrade optoelectronics technologies. This theoretical research investigates a model of the biexciton state in the strongly oblate ellipsoidal quantum dot with the relativistic corrections of mass and Hamiltonian in the framework of the quantum field theory due to the importance of the relativistic effect for this type of quantum dot shapes. The Sturmian function transformation and Wick ordering method to calculate the vacuum state energy eigenvalue of the biexciton system are utilized. Based on the relativistic behavior of interactions, the mass corrections to the Hamiltonian are defined. Dependence of the relativistic mass on the distances between electrons and the constituent mass to the coupling constant is obtained. The results show that as increasing quantum dot size, the relativistic mass and Hamiltonian corrections terms decrease.

Citation: First Author's Full Name, Second Author's Full Name, Third Author's Full Name. The effect of Cinnamon supplementation on lipid profile in response to aerobic exercise. **Journal of Optoelectronical Nanostructures.** ٢٠٢٠; ١ (١): ٤٨-٥٨

Corresponding author: Arezu Jahanshir
Address: Department of Physics and Engineering Sciences, Buein Zahra Technical University, Buein Zahra, Iran. **Tell:** 00982833894492 **Email:** jahanshir@bzte.ac.ir

١.INTRODUCTION

The analytical study on the multi particles in the strongly oblate ellipsoidal semiconductor quantum dots (SOEQD) with relativistic mass correction is an active area of advanced high-technology research [١-٣]. The significance of SOEQD in optoelectrical equipment, communication technology, and quantum information has driven the interest in studying the multi-particle quantum dot based on the relativistic behavior of a system. We choose the multi-particle quantum dots as multi-exciton systems, which play a crucial role in many tasks related to advanced nano-technology also the control of multi-exciton bound states within quantum dots holds great importance for this type of technology. Hence, the theoretical investigation has contributed to understanding the behavior of bound states in quantum optics and nanoelectronics. In this theoretical research, we use biexciton systems in quantum dots and define the energy eigenvalue and relativistic mass corrections based on SOEQD shape [٣-٥]. The characteristics and properties of biexciton as the electrons and holes bound states within the confinement and relativistic effects are investigated. The potential confinement leads to a change in the biexciton constituent particles overlap, and the relativistic effect results in a difference in the constituent mass of particles, which is different from the rest mass and effective mass. Due to the relativistic mass corrections on the energy spectrum, the biexciton system in SOEQD demonstrates a larger emission band, luminescence spectrum, and luminescence decay times compared to non-nano dimensional semiconductors. Multi-excitons in SOEQD have an important property based on face-to-face interactions and lead to multi-particle correlation such as exciton-exciton, biexciton-exciton, and biexciton-biexciton interactions [٤-٦]. These interactions show the new quantum-optical nonlinearities behavior and entangled states [٦-٨]. The measure and features of relativistic mass correction can depend on Multiple elements and factors, including the size and shape of the strongly ellipsoidal semiconductor quantum dots, the composition of the quantum dots material, and the external environment. This theoretical study's results are predicted to offer guidelines for tuning properties of strongly ellipsoidal quantum dots and thus facilitating the advancement of practical nano-dimensional electrical appliances and devices. Consequently, in this research for simplicity of calculations, we choose just biexciton as a four-particle state (electron-hole) of SOEQD and show the relativistic correction effects in the field of nano-quantum physics and optoelectronics [٦-٨]. The remainder of this theoretical investigation is organized in the following format Section ٢, determination of relativistic mass correction in quantum field theory. In Section ٣, a transformation of intertwined spaces is introduced and describes the behavior of SOEQD. In Section ٤, the biexciton mass spectrum and energy eigenvalue of the vacuum state are explained and calculated. Finally, in Section ٥, concluding comments are given.

A. Research Aim

Biexcitons can arise in certain semiconductor quantum dots and nanostructures under specific conditions. We investigate a biexciton Coulomb system in SOEQD, consisting of two excitons bound states (two electron and two holes bound state). Our analytical approach is based on the Sturmian function transformation and Wick ordering method [٩] applied to the quantum oscillator system [٧-١٤]. We determine and calculate the relativistic mass correction to the energy eigenvalue and other properties of a nano-scales $GaAs$ semiconductor that is confined in SOEQD.

B. Sturmian Function and Transformation

One of the mathematical techniques and methods in quantum mechanics and quantum field theory is the Sturmian function transformation [١٤, ١٦]. This technique allows us to describe a quantum system in a Sturmian form simplifying the theoretical analysis and calculation of specific properties. It is widely used in semiconductor quantum dots, wells, and nanostructures. The Sturmian function transformation is beneficial to determine bound state behavior in atomic, molecular, quantum chemistry, solid-state physics, and particle physics. By applying the Sturmian function transformation, the bound state problem can be represented in the second-order linear differential equation that provides a powerful mathematical tool for studying various aspects of nano-scale systems such as wavefunctions, mass spectrum, binding energy and energy eigenvalues, and quantum-optical properties. Hence, we present the Sturmian function transformation to study the biexciton in SOEQD within the relativistic mass correction. The radial Schrödinger equation (RSE) can describe the biexciton properties based on the Sturmian representation and the set of eigenfunctions. The Sturmian function $S_{nl}(r)$ is the solution of RSE

$$\left[-\frac{\hbar^٢}{٢\mu}\frac{d^٢}{dr^٢}+\frac{l(l+١)\hbar^٢}{٢\mu r^٢}++\alpha_{nl}U-E_{\cdot}\right]S_{nl}(r)=\cdot \qquad (١)$$

This function has a notable benefit over RSE when used as a basis for expansion when choosing a different basis of intertwined spaces for the biexciton states. For instance, quantum oscillator wave functions satisfy the needed conditions. The mechanism of transforming the independent parameter has long been a valuable strategy for solving RSE with various types of potentials. In this manuscript, the study of the biexcitons system involves using the Coulomb potential in the strongly ellipsoidal semiconductor quantum dots. A more general equivalence emerges as a change of variable in the wave function of intertwined spaces

$$S_{nl}(r) \approx e^{-a(r)} \approx e^{-r^{١+\sigma}}, r = q^{٢\rho}, \rho = \frac{١}{١+\sigma}, \ \sigma \geq ٠, \qquad (٢)$$

and this transformation can map the nonrelativistic RSE and its solutions for given potentials [١٧, ١٨]. We know that for $r \to \infty$, $a(r)$ can be obtained for certain classes of potentials such as Coulomb or Yukawa-type potentials $\sigma = ٠$. We study the biexciton states within Coulomb potential with a modifiable number of power factors $\sum r^{\alpha}$, and have used a variable change to interpret the Sturmian equation and two intertwined spaces in the context of conventional physical RSE. We also mention that the biexciton states in SOEQD can be addressed using algebraic methods, such as the Wick ordering technique, and solve RSE due to define and determine mass spectrum based on the relativistic mass correction.

٢. THEORETICAL MODEL OF THE ELLIPSOIDAL QUANTUM DOTS

We investigate the four-particle bound state (biexciton) in the ellipsoidal quantum dot with the relativistic mass correction. As we know, ellipsoidal quantum dots have two different types of geometry (prolate and oblate). The strongly prolate ellipsoidal quantum dots explained in the $R^٢$-dimensional space and SOEQD described in the $R^١$-dimensional space [٣-٦]. We will present our theoretical findings on strongly oblate ellipsoidal quantum dots in $GaAs$ semiconductors. It is schematically plotted in Fig. ١. The confinement potential and interaction Coulomb potential of a strongly oblate ellipsoidal quantum dots with condition $a^٢ \gg b^٢$ in $R^٣$ −The dimensional Cartesian coordinate system is considered as follows

$$U_{conf} = \begin{cases} ٠, & \frac{x^٢+y^٢}{a^٢} + \frac{z^٢}{b^٢} \leq ١ \\ \infty, & \frac{x^٢+y^٢}{a^٢} + \frac{z^٢}{b^٢} > ١ \end{cases}, \qquad U_{int} = \sum_{\substack{i,j=١ \\ i \neq j}}^{n} \frac{e_١ e_٢}{٤\pi\varepsilon_r\varepsilon_٠} \frac{١}{|r_i - r_j|} \qquad (٣)$$

where a, b are semiaxes of the ellipsoid, respectively. the Hamiltonian of biexciton bound state (two electrons and two holes) based on the Coulomb potential of interactions U_{int} (as it is shown in Fig.١), in natural unite $\hbar = c = ١$, reads

$$H = \frac{١}{٢}\sum_{i=١}^{n} \frac{p_i^{٢}}{m_i} + U_{int} + U_{conf} \qquad (٤)$$

where e is the charge of particles, ε_r is the relative permittivity of $GaAs$ semiconductor, $\varepsilon_٠$ is the vacuum permittivity, U_{int} corresponds to the electron-hole, hole-hole, and electron-electron interaction. The Biexciton system is a four-body problem that can be constructed in H-type and K-type. H-type coordinates begin by defining electron-electron and hole-hole the relative vector between the centers-of-mass of the biexcitons Fig. ٢.

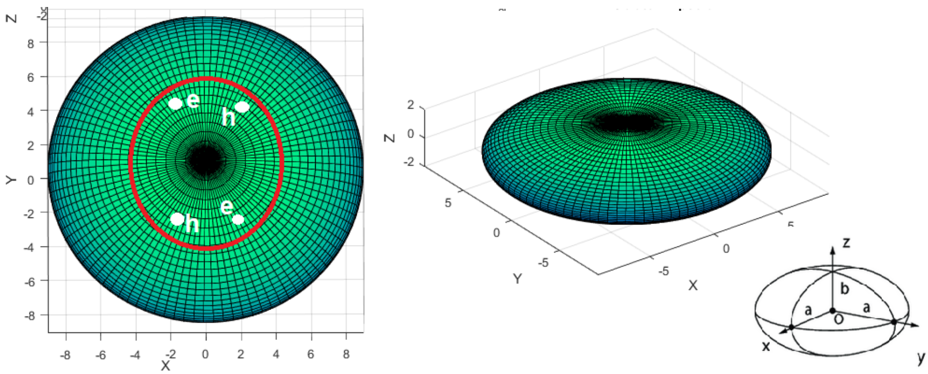

Fig. ١. Schematic plot of SOEQD and Biexciton bound state inside the red cycle.

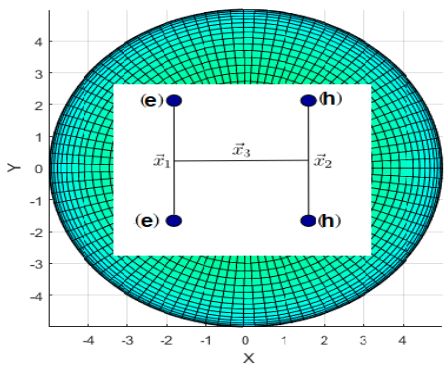

Fig. ٢. Jacobi coordinates $\{x_١, x_٢, x_٣\}$, of biexciton in the four-body problem.

H-channel represents the correlation matrix within two interacting electron-holes, and then the exciton-exciton correlation. K-type coordinates are formed by sequentially establishing the relative vector between the center-of-mass of a subgroup of particles and one extra particle [١٩]. We choose the H-type coordinates to describe biexciton properties in SOEQD. Therefore, the three sets of linearly independent Jacobi coordinates $\{x_١, x_٢, x_٣\}$, for particles in the four-body problem as it is shown in Fig. ٢, reads

Jacobi Coordinate for particle $i = ١$

$$\{x_{١},x_{٢},x_{٣}\} = \left\{r_{٢} - r_{١}, r_{٣} - \frac{m_{١}r_{١}+m_{٢}r_{٢}}{m_{١}+m_{٢}}, r_{٤} - \frac{m_{١}r_{١}+m_{٢}r_{٢}+m_{٣}r_{٣}}{m_{١}+m_{٢}+m_{٣}}\right\}$$

Jacobi Coordinate for Particle $i = ٢$

$$\{x_{١},x_{٢},x_{٣}\} = \left\{r_{١} - r_{٢}, r_{٣} - \frac{m_{١}r_{١}+m_{٢}r_{٢}}{m_{١}+m_{٢}}, r_{٤} - \frac{m_{١}r_{١}+m_{٢}r_{٢}+m_{٣}r_{٣}}{m_{١}+m_{٢}+m_{٣}}\right\}$$

Jacobi Coordinate for Particle $i = ٣$

$$\{x_{١},x_{٢},x_{٣}\} = \left\{r_{٤} - r_{٣}, r_{٢} - \frac{m_{٣}r_{٣}+m_{٤}r_{٤}}{m_{٣}+m_{٤}}, r_{١} - \frac{m_{٢}r_{٢}+m_{٣}r_{٣}+m_{٤}r_{٤}}{m_{٢}+m_{٣}+m_{٤}}\right\}$$

Jacobi Coordinate for Particle $i = ٤$

$$\{x_{١},x_{٢},x_{٣}\} = \left\{r_{٣} - r_{٤}, r_{٢} - \frac{m_{٣}r_{٣}+m_{٤}r_{٤}}{m_{٣}+m_{٤}}, r_{١} - \frac{m_{٢}r_{٢}+m_{٣}r_{٣}+m_{٤}r_{٤}}{m_{٢}+m_{٣}+m_{٤}}\right\} \quad (٥)$$

and

the position vector of Particle $i = ١$

$$\{r_{١}\} = \left\{\frac{m_{٢}}{m_{١}+m_{٢}}x_{١} + \frac{m_{٣}+m_{٤}}{m_{١}+m_{٢}+m_{٣}+m_{٤}}x_{٣} + R_c\right\}$$

the position vector of Particle $i = ٢$

$$\{r_{٢}\} = \left\{-\frac{m_{١}}{m_{١}+m_{٢}}x_{١} + \frac{m_{٣}+m_{٤}}{m_{١}+m_{٢}+m_{٣}+m_{٤}}x_{٣} + R_c\right\}$$

the position vector of Particle $i = ٣$

$$\{r_{٣}\} = \left\{\frac{m_{٤}}{m_{٣}+m_{٤}}x_{٢} - \frac{m_{١}+m_{٢}}{m_{١}+m_{٢}+m_{٣}+m_{٤}}x_{٣} + R_c\right\}$$

the position vector of Particle $i = ٤$

$$\{r_{٤}\} = \left\{-\frac{m_{٣}}{m_{٣}+m_{٤}}x_{٢} - \frac{m_{١}+m_{٢}}{m_{١}+m_{٢}+m_{٣}+m_{٤}}x_{٣} + R_c\right\} \quad (٦)$$

where r_i represents the position vector of the i-particle with the rest mass m_i, and R_c is the coordinate of the center of mass. Hence Eq. (٤) reads

$$H = \frac{١}{٢}\sum_{\substack{i=١ \\ j=١...٣}}^{٤} \frac{p_{x_j}^{٢}}{m_i} + \sum_{\substack{i,j=١ \\ i\neq j}}^{٤} \frac{e^{٢}}{٤\pi\varepsilon_r\varepsilon.}\frac{١}{|r_i-r_j|} =$$

$$\frac{m_{١}+m_{٢}}{٢m_{١}m_{٢}}p_{x_{١}}^{٢} + \frac{m_{٣}+m_{٤}}{٢m_{٣}m_{٤}}p_{x_{٢}}^{٢} + \frac{m_{١}+m_{٢}+m_{٣}+m_{٤}}{٢m_{١}m_{٢}m_{٣}m_{٤}}p_{x_{٣}}^{٢} + \frac{e^{٢}}{٤\pi\varepsilon_r\varepsilon.}\frac{١}{|x_{١}|} -$$
$$\frac{e^{٢}}{٤\pi\varepsilon_r\varepsilon.}\frac{١}{\left|\frac{\mu_{١}x_{١}}{m_{١}} - \frac{\mu_{٢}x_{٢}}{m_{٤}} + x_{٣}\right|} - \frac{e^{٢}}{٤\pi\varepsilon_r\varepsilon.}\frac{١}{\left|\frac{\mu_{١}x_{١}}{m_{١}} + \frac{\mu_{٢}x_{٢}}{m_{٤}} + x_{٣}\right|} - \frac{e^{٢}}{٤\pi\varepsilon_r\varepsilon.}\frac{١}{\left|\frac{\mu_{١}x_{١}}{m_{٢}} + \frac{\mu_{٢}x_{٢}}{m_{٣}} - x_{٣}\right|} -$$
$$\frac{e^{٢}}{٤\pi\varepsilon_r\varepsilon.}\frac{١}{\left|\frac{\mu_{١}x_{١}}{m_{٢}} - \frac{\mu_{٢}x_{٢}}{m_{٣}} - x_{٣}\right|} + \frac{e^{٢}}{٤\pi\varepsilon_r\varepsilon.}\frac{١}{|x_{٢}|} \quad (٧)$$

Now for simplicity, we consider $m_{١} = m_{٢} = m_e^*$ and $m_{٣} = m_{٤} = m_h^*$ in the electrostatic field, and

$$m_e^* = \alpha m_e, m_h^* = \beta m_e, \ k = \sqrt{\frac{\alpha}{\beta}}, \ x_{١} = \frac{R_{١}}{\sqrt{\alpha m_e}}, x_{٢} = \frac{R_{٢}}{\sqrt{\beta m_e}}, x_{٣} = \frac{R_{٣}}{٢\sqrt{\alpha m_e}}$$

Then Eq. (٧) is realized by the following parameters

$$H = \frac{١}{٤m_١} p_{x_١}^{٢} + \frac{١}{٤m_٢} p_{x_٢}^{٢} + \frac{m_١+m_٢}{m_١m_٢} p_{x_٢}^{٢} + \frac{\sqrt{m_١}e^{٢}}{٤\pi\varepsilon_r\varepsilon_٠}\frac{١}{|R|} - \frac{\sqrt{m_١}e^{٢}}{٢\pi\varepsilon_r\varepsilon_٠}\frac{١}{|(R+r)-kz|} - \frac{\sqrt{m_١}e^{٢}}{٢\pi\varepsilon_r\varepsilon_٠}\frac{١}{|(R+r)+kz|} - \frac{\sqrt{m_١}e^{٢}}{٢\pi\varepsilon_r\varepsilon_٠}\frac{١}{|(R-r)+kz|} - \frac{\sqrt{m_١}e^{٢}}{٢\pi\varepsilon_r\varepsilon_٠}\frac{١}{|(R-r)-kz|} + \frac{\sqrt{m_١}e^{٢}}{٤\pi\varepsilon_r\varepsilon_٠}\frac{١}{|kz|} \quad (٨)$$

We assume that the motion of the electrons and holes clusters are limited by themselves and do not move freely in quantum dots. These clusters move only in the biexcitonic structure, hence

$$x_١ = const \Longrightarrow p_{x_١} = ٠, x_٢ = const \Longrightarrow p_{x_٢} = ٠$$

therefore, the radial part of RSE with Hamiltonian Eq. (٨) in the cylindrical coordinate system (ρ,z,φ) presents as follows

$$(H - E)\Psi(R,r) = ٠ \Longrightarrow$$

$$\left(\frac{m_e^*+m_h^*}{m_e^*m_h^*} p_{R_٢}^{٢} - \frac{\sqrt{m_e^*}e^{٢}}{\pi\varepsilon_r\varepsilon_٠}\left[\frac{١}{|R_١+R_٢|} + \frac{١}{|R_١-R_٢|}\right] - E\right)\Psi(R,r) = ٠ \quad (٩)$$

Where $\mathfrak{M}$ is the magnetic quantum number and the wavefunction has the form $\Psi(R_١, R_٢) = \frac{١}{\pi^{\frac{١}{٢}}} e^{i\mathfrak{M}\varphi}\Phi(R;\rho,z)$. Now, based on SOEQD $z = ٠$. And finally, Eq. (٢١) will have the following form

$$\left(\begin{array}{l} -\frac{m_e^*+m_h^*}{٢m_e^*m_h^*}\left[\frac{\partial^{٢}}{\partial\rho^{٢}} + \frac{١}{\rho}\frac{\partial}{\partial\rho} - \frac{m^{٢}}{\rho^{٢}}\right] - \\ -\frac{\sqrt{m_e^*}e^{٢}}{\pi\varepsilon_r\varepsilon_٠}\left[\frac{١}{|R_١+R_٢|} + \frac{١}{|R_١-R_٢|}\right] - E \end{array}\right)\Phi(R_١, R_٢;\rho,z) = ٠ \quad (١٠)$$

As we have two clusters, we choose $\rho = ٢\sqrt{\rho_١\rho_٢}$, and after some mathematical calculation, the Hamiltonian of relative motion of exciton-exciton interaction as a biexciton bound state with the radius vectors $\rho_١$, $\rho_٢$ in SOEQD reads

$$\left(\begin{array}{l} -\frac{m_e^*+m_h^*}{٢m_e^*m_h^*}\left[\left(\rho_١\frac{\partial^{٢}}{\partial\rho_١^{٢}} + \rho_٢\frac{\partial^{٢}}{\partial\rho_٢^{٢}}\right) + \left(\frac{\partial}{\partial\rho_١} + \frac{\partial}{\partial\rho_٢}\right) - \mathfrak{M}_١^{٢}\left(\frac{١}{٤\rho_١^{٢}} + \frac{١}{٤\rho_٢^{٢}}\right)\right] - \\ \frac{\sqrt{m_e^*}e^{٢}}{\pi\varepsilon_r\varepsilon_٠}\left[\frac{(\rho_١+\rho_٢)}{\sqrt{(\rho_١+\rho_٢)^{٢}+R_١^{٢}}} + \frac{(\rho_١+\rho_٢)}{\sqrt{(\rho_١+\rho_٢)^{٢}-R_١^{٢}}}\right] - (\rho_١+\rho_٢)E \end{array}\right)\Phi(R_١,\rho_١,\rho_٢) = \quad (١١)$$

٣. AUXILIARY SYMPLECTIC SPACE IN QUANTUM THEORY

In the context of quantum mechanics, an auxiliary symplectic space refers to an additional space that is introduced to simplify the calculation and analysis of a system. This subject deals with symplectic manifolds, which are mathematical objects that capture the geometric properties of dynamical quantum systems, and often are described using position and momentum variables [۲۰, ۲۱]. This method is essential and beneficial in solving the RSE of the bound states. It allows different aspects of the bound state's properties and characteristics to be presented and clarified in the position and momentum spaces. By introducing an auxiliary symplectic space as a powerful mathematical tool and technique, one can provide a clearer understanding of the dynamics of bound states. Dimension of the auxiliary symplectic space depends on the particular bound states and the goals of the investigations. Now, we introduce the auxiliary symplectic space in quantum theory for solving the mass spectra and energy eigenvalues of the biexciton bound state using Eq. (۱۵) in SOEQD. For this reason, we identify canonical operators in Eq. (

(۱۱), by changing $r = q^{۲}$, and always map $r = ۰$ into $q = ۰$ and map $r = \infty$ into $q = \infty$. Two intertwined spaces can be modified by connections

$$\frac{d^{۲}}{dr^{۲}} = \frac{-۱}{۴q}\left(\frac{۱}{q^{۲}}\frac{d}{dq} - \frac{۱}{q}\frac{d^{۲}}{dq^{۲}}\right) \text{ and } \frac{d}{dr} = \frac{۱}{۲q}. \quad (۱۲)$$

Hence, the radial Laplacian in a $N = ۳$ dimensional Riemannian space to the intertwined space

$$\Delta_r = \frac{d^{۲}}{dr^{۲}} + \frac{N-۱}{r}\frac{d}{dr} \to \Delta_q = \frac{d^{۲}}{dq^{۲}} + \frac{\mathcal{D}-۱}{q}\frac{d}{dq}, \quad (۱۳)$$

is transformed, and $\Phi(R_۱,\rho_۱,\rho_۲) \to \chi(q_۱^{۲},q_۲^{۲})$. Therefore, under the relative motion of particles in the biexciton bound state, to determine the energy eigenvalue and mass spectra of biexciton in SOEQD, we change the variables $\rho_۱ = q_۱^{۲}$, $\rho_۲ = q_۲^{۲}$ and define the wavefunction

$$\Phi(R_۱,\rho_۱,\rho_۲) = q_۱^{|\mathfrak{W}|} q_۲^{|\mathfrak{W}|} \chi(q_۱^{۲},q_۲^{۲}), \quad (۱۴)$$

and then Eq. (

(۱۱)

reads

$$\left(-\frac{m_e^*+m_h^*}{۲m_e^*m_h^*}\left[\left(\frac{\partial^۲}{\partial q_۱^۲}+\frac{\partial^۲}{\partial q_۲^۲}\right)+(۲|\mathfrak{W}|+۱)\left(\frac{۱}{q_۱}\frac{\partial}{\partial q_۱}+\frac{۱}{q_۲}\frac{\partial}{\partial q_۲}\right)\right]-\frac{۴\sqrt{m_e^*}e^۲}{\pi\varepsilon_r\varepsilon_۰}\left[\frac{(q_۱^۲+q_۲^۲)}{\sqrt{(q_۱^۲+q_۲^۲)^۲+R_۱^۲}}+\frac{(q_۱^۲+q_۲^۲)}{\sqrt{(q_۱^۲+q_۲^۲)^۲-R_۱^۲}}\right]-۴(q_۱^۲+q_۲^۲)E\right)\chi(R_۱,q_۱^۲,q_۲^۲)=۰ \quad (۱۵)$$

where $\mathcal{D}=۲(۱+|\mathfrak{W}|)$ is the dimension of the auxiliary intertwined space. The biexciton bound state as a quantum oscillating system can be presented by the Wick ordering technique in the auxiliary symplectic space. It formulates the momentum ($\hat{p}$) and position ($\hat{q}$) operators (canonical variables [۲۱]) in terms of

$$\hat{a}^+=\frac{\sqrt{۲}}{۲}\left(\sqrt{\omega}\hat{q}-\frac{i}{\sqrt{\omega}}\frac{d}{dq}\right) \text{ and } \hat{a}=\frac{\sqrt{۲}}{۲}\left(\sqrt{\omega}\hat{q}+\frac{i}{\sqrt{\omega}}\frac{d}{dq}\right) \quad (۱۶)$$

operators, where $\hat{q}=\sqrt{۲\omega}(\hat{a}^++\hat{a})$, $\frac{d}{dq}=i\sqrt{\frac{\omega}{۲}}(\hat{a}^+-\hat{a})$, and ω is the vacuum state frequency of biexciton. Substituting operators $\hat{q}$ and $\hat{p}=\frac{d}{dq}$ into the Eq. (۱۵) and ordering by operators $\hat{a}^+,\hat{a}$. The Wick ordering product [۸]over the canonical operators is $\hat{p}^۲\approx\omega\frac{\Gamma\left(\frac{\mathcal{D}}{۲}+۱\right)}{\Gamma\left(\frac{\mathcal{D}}{۲}\right)}+:\hat{p}^۲:$, and $\hat{q}^{۲u}\approx\frac{۱}{\omega^u}\frac{\Gamma\left(\frac{\mathcal{D}}{۲}+u\right)}{\Gamma\left(\frac{\mathcal{D}}{۲}\right)}+:\hat{q}^{۲u}:$, then the interaction Hamiltonian of four particles bound state is realized by the following equation

$$H=\omega(\hat{a}^+\hat{a})+\frac{\mathcal{D}}{۴}\omega+\int\left(\frac{dk}{۲\pi}\right)^{\mathcal{D}}\breve{W}(k^۲)e^{\frac{-k^۲}{۴\omega}}:e^{ik\hat{q}}:-\frac{\omega^۲}{۲}\left(:\hat{q}^۲:+\frac{\mathcal{D}}{۲\omega}\right) \quad (۱۷)$$

Where $W(q^۲)\int\left(\frac{dk}{۲\pi}\right)^{\mathcal{D}}\breve{W}(k^۲)e^{ikq}$ is the potential, $H_۰=\omega(\hat{a}^+\hat{a})$ is he Hamiltonian of the free oscillator, $\varepsilon_۰=\frac{\mathcal{D}\omega}{۴}+\int\left(\frac{dk}{۲\pi}\right)^{\mathcal{D}}\breve{W}(k^۲)e^{\frac{-k^۲}{۴\omega}}$ is the energy of the vacuum or the ground state energy, and $H_I=\int\left(\frac{dk}{۲\pi}\right)^{\mathcal{D}}\breve{W}(k^۲)e^{\frac{-k^۲}{۴\omega}}:e^{ik\hat{q}}:$ is the interaction Hamiltonian. :∗: is the normal ordering (Wick ordering[۸]) symbol and we consider that in the symplectic intertwined spaces the interaction Hamiltonian doesn't contain $:\hat{q}^۲:$ and $\hat{q}^۲$. These terms are included in the vacuum state energy. Hence, we define a specific condition that determines the frequency of the biexciton bound state $\omega^{٢} + \int \left(\frac{dk}{٢\pi}\right)^{\mathcal{D}} \left(\frac{k^{٢}}{N}\right) e^{\frac{-k^{٢}}{٤\omega}} \breve{W}\left(k^{٢}\right) = ٠$. Now, using conditions $\varepsilon_{٠} = ٠$ and $\frac{d\varepsilon_{٠}}{d\omega} = ٠$ for creating the bound state (i.e., the confined state exists at the lowest point of oscillator frequency and energy eigenvalue) [١٧], one can define the ground state energy eigenvalue and mass spectrum of biexciton system in SOEQD.

٤. RELATIVISTIC CORRECTIONS

Increasing the Coulomb interaction coupling constant in semiconductor quantum dots can lead to various properties, such as relativistic behavior of interaction behavior. By decreasing the size and shape of the semiconductor quantum dots the electrons and holes become more closely, leading to a stronger Coulombic coupling constant. By increasing the number of electrons, holes or both the carrier density within the semiconductor quantum dot Coulombic coupling constant can be enhanced. By choosing materials or modifying the dielectric medium based on connecting two or more semiconductor layers the strength of the Coulomb interaction can be increased. Applying a strong electromagnetic field on a system the coupling constant strength can be increased. Hence, we choose theoretical modification to increase the Coulomb interaction in the strong oblate ellipsoidal $GaAs$ quantum dot and describe the relativistic conditions on the biexciton bound state. The mass spectrum and energy eigenvalue of the bound states in SOEQD are present based on the quantum mechanics approaches and Feynman path integral method in the quantum field theory [٢٢-٢٦]. As we know, the long-range and small-range behaviors of the propagator function of the related system's current with the different quantum numbers can determine the main properties of the bound states in SOEQD. The Feynman functional path integral method allows us to average over the field and describe the Green function $G(r - r')$ of bound state and interactions of particles in quantum field theory, similar to the Schrödinger picture in quantum physics. We explain the related current of charged particles in the system and obtained current through averaging over the external field for bounded particles [٢٣, ٢٤, ٢٦]. This method defines the kernel function of two charged particles in SOEQD with the specific rest masses. Hence, one can determine the n-point function by averaging over the field $\Pi(r - r') = \langle G_{١}(r - r')G_{٢}(r - r') \cdots G_n(r - r')\rangle_{\boldsymbol{A}}$. The Green function $G_m(x)$ for the scalar particle with the rest mass m and coupling constant g in the external field $\boldsymbol{A}(x)$ is defined from

$$\left[\left(i\frac{\partial}{\partial x_\alpha} + \frac{g}{c\hbar}\boldsymbol{A}(x)\right)^{٢} + \frac{c^{٢}m^{٢}}{\hbar^{٢}}\right] G_m(x) = \delta(x). \quad (١٨)$$

The Green function in the form of path integral can be written as

$$G_m(r,r'|\boldsymbol{A}) = \int_{۰}^{\infty} \frac{d\mu}{(۴\pi\mu)^{۲}} \boldsymbol{e}^{\left\{-\mu m^{۲} - \frac{(r-r')^{۲}}{۴\mu}\right\}} \times \int d\sigma_\xi \boldsymbol{e}^{\left\{ig\int_{۰}^{۱} d\xi \frac{dZ(\xi)}{d\xi}\boldsymbol{A}(\xi)\right\}} \qquad (۱۹)$$

where

$Z(\xi) = (r-r')\xi + r' - ۲\sqrt{\mu}B(\xi)$, $d\sigma_\xi = N\delta B(\xi)\boldsymbol{e}^{\left\{-۰.۵\int_{۰}^{۱} d\xi\,(B(\xi))^{۲}\right\}}$, and N is the normalization condition with $\int d\sigma_\xi = ۱$ with boundary conditions $B(۰) = B(۱) = ۰$. Then using the variational method and Eq. (۱۹) and $\Pi(r-r') = \langle G_۱(r-r')G_۲(r-r')\cdots G_n(r-r')\rangle_{\boldsymbol{A}}$, the polarization function of n-point function $\Pi(r-r')$ reads

$$\Pi(r-r') = \int\cdots\int_{۰}^{\infty} \frac{d\mu_۱\cdots d\mu_n}{(۸\pi^{۲}|r-r'|)^n}\, \Omega(\mu_۱,\cdots,\mu_n)\boldsymbol{e}^{\left\{-\frac{|r-r'|}{۲}\left[\left(\frac{m_۱^{۲}}{\mu_۱}+\mu_۱\right)+\cdots+\left(\frac{m_n^{۲}}{\mu_n}+\mu_n\right)\right]\right\}} \qquad (۲۰)$$

where

$$\Omega(\mu_۱,\cdots,\mu_n) = c_۱\cdots c_n\int\int d\mu_۱\cdots d\mu_n\,\boldsymbol{e}^{\left\{-\int_{۰}^{\alpha} d\tau\sum_{k=۱}^{n} ۰.۵\mu_k v_k^{۲} - \left[(U_{۱۱}+\cdots+U_{nn}) - ۲\sum_{\substack{i,j=۱\\ i\neq j}}^{n} U_{ij}\right]\right\}} \qquad (۲۱)$$

the functional $\Omega(\mu_۱,\cdots,\mu_n)$ resembling the behavior of $n-$ particles with masses $\mu_۱,\cdots,\mu_n$ in the non-relativistic quantum mechanics and with the potential and nonpotential interaction that includes in U_{ij} in the form of the Feynman path integral and at $|r-r'|\to ۰$, it is analogous to the relation

$$\Omega(\mu_۱,\cdots,\mu_n) \cong \boldsymbol{e}^{\{-|r-r'|\mathrm{E}((\mu_۱,\cdots,\mu_n))\}} \qquad (۲۲)$$

where $E(\mu_۱,\cdots,\mu_n)$ is an eigenvalue of $HR(r) = ER(r)$ for the motion of $n-$ bounded particles with masses $\mu_۱,\cdots,\mu_n$, it depends on the parameters $\mu_۱,\cdots,\mu_n$, $m_۱,\cdots,m_n$, g, the rest mass, the constituent mass, and the coupling constant, respectively. The constituent masses of interacting particles mean the mass of particles inside the bounded system. Then, the Green's function at $|r-r'|\to ۰$ can determine the bound state total mass M of $n-$ bounded particles as follows [۲۵, ۲۷]

$$\langle G_m(r,r'|\boldsymbol{A})\rangle \geqslant \frac{const}{|r-r'|}\boldsymbol{e}^{\{-M|r-r'|\}} \qquad (۲۳)$$

Using Eq. (۷) and relation $\Pi(r-r') = \langle G_۱(r-r')G_۲(r-r')\cdots G_n(r-r')\rangle_{\boldsymbol{A}}$ one can demonstrate the polarization function $\Pi(r-r') \cong \boldsymbol{e}^{\{-M|r-r'|\}}$. Hence, the bound state mass spectrum is defined as $M = \lim_{|r-r'|\to\infty}\frac{-\ln\Pi(r-r')}{|r-r'|}$. We can determine the bound state mass M based on the method of steepest descent in the following manner for ($i = ۱ \cdots n$)

$$M = \frac{\partial}{۲\partial \mu_i}\left(\frac{\mu_۱ m_۱^۲+\cdots+\mu_n m_n^۲}{\mu_۱\ldots\mu_n} + \mu_۱ + \cdots + \mu_n + ۲\mathrm{E}(\mu_۱,\cdots\mu_n)\right) \quad (۲۴)$$

where

$$\mu_i = \frac{m_i^۲}{\mu_i} + ۲\mu_i \frac{\partial}{\partial \mu_i}\mathrm{E}(\mu_۱,\cdots\mu_n) = \sqrt{m_i^۲ - ۲\mu^۲ \frac{\partial}{\partial \mu_i}\mathrm{E}(\mu_۱,\cdots\mu_n)} \quad (۲۵)$$

with($i = ۱, ۲, \cdots, n$), $\mu = \frac{\mu_۱\mu_۲,\cdots\mu_n}{\mu_۱+\mu_۲+,\cdots+\mu_n}$ is the reduced mass of the system, μ_i is the constituent mass of particles, which determines the relativistic mass correction of each particle [۱۷, ۱۸, ۲۷]. Now, we define the relativistic correction on the Hamiltonian of biexciton bound state. The total Hamiltonian of the biexciton can be further developed with the guidance of relativistic effects such as the relativistic correction to the mass, kinetic energy, spin interaction, and other perturbative terms. The relativistic effects result from a change in the biexciton bound states' energy brought on by relativistic mass. The other one is spin interaction, which is contained in spin-spin, spin-orbit, and spin tensor interaction. Coupling [۲۷, ۲۸]. These relativistic corrections affect the ground and excited energy levels of the biexciton in SOEQD, which have a considerable effect and meaningful contribution to the electrical and optical characteristics of quantum dots. The total Hamiltonian $H = H_۰ + H_{SS} + H_{LS} + H_T + H_{rel} + H_{pert}$ describes all interaction potentials [۲۹] that are included in the term U_{ij} of Eq. (۲۱), and all energy spectrum reads $\varepsilon_{tot} = \varepsilon_۰ + \varepsilon_{SS} + \varepsilon_{LS} + \varepsilon_T + \varepsilon_{rel} + \varepsilon_{pert}$. The potential of interaction has the form

$$U_{ij} = \frac{(-۱)^{i+j} g^۲}{۲}\int_۰^t\int_۰^t d\tau_۱ d\tau_۲\, \dot{Z}_\alpha(\tau_۱) D_{\alpha\beta}\left(Z^i(\tau_۱) - Z^j(\tau_۲)\right)\dot{Z}_\beta(\tau_۲) =$$

$$\frac{(-۱)^{i+j} g^۲}{۲}\int_۰^t\int_۰^t d\tau_۱ d\tau_۲ \left(\frac{\vec{r}}{|r|} + \frac{\dot{\vec{r}}_i(\tau)}{c}\right)\left(\frac{\vec{r}}{|r|} + \frac{\dot{\vec{r}}_j(\tau)}{c}\right)\int \frac{dq}{(۲\pi)^۳}\int_{-\infty}^{+\infty}\frac{ds}{۲\pi} D_{\alpha\beta}\left(\vec{q}^۲ + \frac{s^۲}{c^۲}\right) \times$$

$$\times\, \boldsymbol{e}^{\left\{is\tau + \frac{is}{c}\left(r_i^{\cdot}(\tau_۱) - r_j^{\cdot}(\tau_۲)\right) + i\vec{q}\left(\vec{r}_i(\tau_۱) - \vec{r}_j(\tau_۲)\right)\right\}} = U_{ij}{}^۱ + U_{ij}{}^۲ \quad (۲۶)$$

The term $U_{ij}{}^۱$ corresponds to the contribution of one photon exchange, and the term $U_{ij}{}^۲$ gives only cross term. c is the speed of light, and α is the coupling constant of one photon exchange. We suppose that in the initial moment, particles are in the rest and interact among themselves only by electrostatic field. In this case, we have $\Theta_{ij} = ۱ + \frac{\vec{r}}{|r|c}\left(\dot{\vec{r}}_i(\tau_۱) - \dot{\vec{r}}_j(\tau_۲)\right) + \frac{۱}{c^۲}\left(\dot{\vec{r}}_i(\tau_۱)\dot{\vec{r}}_j(\tau_۲)\right) = ۱$. From $r^\mu = (ct, \vec{r}), \tau \equiv it, \vec{r} = \vec{r}_i(\tau_۱) - \vec{r}_j(\tau_۲)$, it follows that the asymptotic behavior of U_{ij} must depend on t in a linear way [۳۰]. According to this, the dependence of $\dot{r}$ is Euclidean time on τ is the proper time of the composite particle is chosen as $\dot{r} = \dot{r}_{۱}(\tau_{۱}) - \dot{r}_{۲}(\tau_{۲}) = c(\tau_{۱} - \tau_{۲}) = c\tau$, $\tau_{۱}$ and $\tau_{۲}$. After performing a series of mathematical calculations and substitutions, we get the nonperturbative relativistic correction on the total Hamiltonian as a relativistic correction term which is defined by the equation

$$H_{rel} = U_v(r)\left[\left(۱ + \frac{\ell(\ell+۱)}{(۲\hbar c r\mu)^{۲}}\right)^{-\frac{۱}{۲}} - ۱\right] \qquad (۲۷)$$

where μ is the reduced mass of biexciton, r is the distance between the clusters, $U(r) = U_v(r) + U_s(r)$ is the potential interaction, $U_s(r)$ is the scalar term and $U_v(r)$ is the vector term of potential interaction of the biexciton SOEQD [۲۹]. The non-relativistic results of confined states are no longer sufficient for the description of the Large Hadron Collider, experimental results in particle physics, and nano-scale interactions. Therefore, it is necessary to consider the relativistic corrections on mass, energy, spins, and nonperturbative and perturbative interactions. As we are aware, the real relativistic corrections are very small, therefore within the framework of quantum field theory, the theoretical approach outlined in Eq. (۲۷) simplifies deriving the relativistic corrections for a nonrelativistic interaction potential. Hence, the relativistic Hamiltonian of biexciton with the mass $m_{ext} = \frac{۱}{m_e^*} + \frac{۱}{m_h^*}$, in the strongly oblate ellipsoidal $GaAs$ quantum dot within the Coulomb interaction in the natural unit has the form $H_{rel} = -\frac{\sqrt{m_{ext}}}{\varepsilon_r a_B m_e r}\left[\left(۱ + \frac{\ell(\ell+۱)}{(۲r\mu)^{۲}}\right)^{-\frac{۱}{۲}} - ۱\right]$, a_B is the Bohr radius.

۵. RESULTS AND DISCUSSION

One of the main results and interesting effects of SOEQD is the potential confinement of particles, which is achieved through photon exchange. This potential confinement leads to distinct electrical and optical properties that are important and valuable in quantum-optical devices, nano-electronic components, and optoelectronic equipment. As it is widely recognized, the optical properties of SOEQD are an essential aspect of nanophysics. Therefore, exciton and biexciton states in SOEQD are an interesting subject. As we show in previous sections the relativistic corrections on the total Hamiltonian can increase the accuracy of results in calculating energy, mass, and spin interactions. In this section, we will calculate the relativistic corrections to the mass and Hamiltonian of biexciton in SOEQD, which are performed theoretically. Two-center approximation method is a universal method to describe the biexciton bound states of the electron-hole (exciton) interacting clusters. This tool simplifies system interactions by treating the clusters independently. This approximation assumes that the biexciton bound state's wavefunction can be explained as separate wavefunctions for each excitonic cluster. The biexciton properties can be solved using RSE or the relativistic Bethe-Salpeter equation. This approach allows us to describe the biexciton analytically and numerically, compared to considering the interactions between and within the excitonic bound states. Now, to determine the biexciton mass spectrum and properties, we use the theoretical effective mass of an exciton in a $GaAs$ quantum dot. It typically considers the reduced mass of the exciton (electron-hole) system $m_{ext} = \frac{١}{m_e^*} + \frac{١}{m_h^*}$. Once m_{ext} theoretically obtained, it can be used to describe the exciton dynamics within the quantum dot system using the two-center approximation method. Therefore, due to the present relativistic behavior of the system, we present the method based on quantum mechanics and quantum field theory approaches. Hence, using the Bethe-Salpeter equation $\varepsilon_{.}(R_١, q_١^٢, q_٢^٢) = (H - E)\chi(R_١, q_١^٢, q_٢^٢) = ٠$ and Eq. (١٥) we will describe relativistic processes and the behavior of the system within the strong Coulomb interaction with the coupling constant α. Relativistic effects are accounted for by including relativistic energy and expanding the term $\sqrt{m^٢ + p_r^٢} \approx \min_{\mu}\left(\frac{\mu}{٢} + \frac{m^٢ + p_r^٢}{٢\mu}\right)$, hence $\varepsilon_{.}(R_١, q_١^٢, q_٢^٢) = \varepsilon_{.}(\mu)$, $\chi(R_١, q_١^٢, q_٢^٢) = \chi(\mu)$ in the axillary symplectic space is obtained in the following way $\varepsilon_{.}(\mu) = \frac{\mathcal{D}\omega}{٤\mu} + ٤q^٢(U(q) - E) = ٠$, and then Eq. (١٥) reads

$$\varepsilon_{.}(\mu)\chi(\mu) = \left(\frac{\mathcal{D}}{٤\mu}(\omega_١ + \omega_٢) - ٨\alpha\sqrt{m_{ext}}\left(\frac{\mathcal{D}}{٢\omega_١} + \frac{\mathcal{D}}{٢\omega_٢}\right)\left[\left(\left(\frac{\mathcal{D}}{٢\omega_١} + \frac{\mathcal{D}}{٢\omega_٢}\right)^٢ + R_١^٢\right)^{-\frac{١}{٢}}\right] - ٤E\left(\frac{\mathcal{D}}{٢\omega_١} + \frac{\mathcal{D}}{٢\omega_٢}\right)\right)\chi(\mu) = ٠ \qquad (٢٨)$$

where $\frac{١}{m_{ext}} = \left(\frac{١}{m_e^*} + \frac{١}{m_h^*}\right)$ is considered as the rest mass of (exciton) electron-hole confined state in the two-point approximation method, $\mu = \frac{\mu_{ext}^*}{٢}$ is the reduced mass of biexciton and μ_{ext}^* is the constituent mass of exciton (electron-hole), and $\alpha = \frac{١}{٤\pi\varepsilon_r}$ in the natural unit. These masses describe the mass of the electron and hole inside the biexciton bound state and differ from the rest mass and effective mass of particles in the strongly oblate ellipsoidal $GaAs$ quantum dot. We consider using conditions $\frac{d\varepsilon_{\cdot}(\mu)}{d\omega_{١}} = ٠$ and $\frac{d\varepsilon_{\cdot}(\mu)}{d\omega_{٢}} = ٠$, and $\omega_{١}$, $\omega_{٢}$ are oscillator frequencies of clusters in a biexciton-bound state (for more detail see [٣١]). We determine oscillator frequencies for conditions $R_{١} = ٠$ and $R_{١} = \infty$ and define $\omega_{+} = \omega_{١} = \omega_{٢}$, and Eq. (٢٨) reads

$$\left(\frac{\mathcal{D}\omega_{+}}{٢\mu} - ٨\alpha\sqrt{m_{ext}}\,\frac{\mathcal{D}}{\omega_{+}}\left[\left(\left(\frac{\mathcal{D}}{\omega_{+}}\right)^{٢} + R_{١}^{٢}\right)^{-\frac{١}{٢}}\right] - ٤E\,\frac{\mathcal{D}}{\omega_{+}}\right)\chi(\mu) = ٠ \qquad (٢٩)$$

utilizing and applying correlations

$$\left(\left(\frac{\omega_{+}R_{١}}{\mathcal{D}}\right)^{٢} + ١\right)^{-\frac{١}{٢}} \approx \boldsymbol{e}^{-\left(\frac{\omega_{+}R_{١}}{\mathcal{D}}\right)^{٢}} + \frac{١}{٢}\left(\frac{\omega_{+}R_{١}}{\mathcal{D}}\right)^{٢} \qquad (٣٠)$$

and Eq. (٢٩) reads

$$\left(\frac{\mathcal{D}\omega_{+}}{٢\mu} - ٨\alpha\sqrt{m_{ext}}\left(\boldsymbol{e}^{-\left(\frac{\omega_{+}R_{١}}{\mathcal{D}}\right)^{٢}} + \frac{١}{٢}\left(\frac{\omega_{+}R_{١}}{\mathcal{D}}\right)^{٢}\right) - ٤E\,\frac{\mathcal{D}}{\omega_{+}}\right)\chi(\mu) = ٠ \qquad (٣١)$$

and then, considering that the distance between two clusters is $R_{١}, (R_{١} \ll ١)$, where $R_{١}||z$, after performing specific mathematical substitutions, the energy eigenvalue can be defined from Eq. (٣١) in the form

$$E = \frac{\omega_{+}^{٢}}{٨\mu} - \frac{٨\alpha\sqrt{m_{ext}}}{\mathcal{D}}\omega_{+}\boldsymbol{e}^{-\left(\frac{\omega_{+}R_{١}}{\mathcal{D}}\right)^{٢}} + \frac{٤\alpha R_{١}^{٢}\sqrt{m_{ext}}}{\mathcal{D}^{٣}}\omega_{+}^{٣} \qquad (٣٢)$$

Then, taking advantage of the condition $\omega_{+}\frac{d\varepsilon_{\cdot}(\mu)}{d\omega_{+}} = ٠$, by minimizing the Eq. (٣٢), we define the equation for describing oscillator frequency ω_{+} of biexciton in strongly oblate ellipsoidal $GaAs$ quantum dot

$$\frac{dE}{d\omega_{+}} = \frac{\omega_{+}}{٤\mu} + \frac{٨\alpha R_{١}^{٢}\sqrt{m_{ext}}}{\mathcal{D}^{٣}}\omega_{+}^{٢}\,\boldsymbol{e}^{-\left(\frac{\omega_{+}R_{١}}{\mathcal{D}}\right)^{٢}} - \frac{٤\alpha\sqrt{m_{ext}}}{\mathcal{D}}\boldsymbol{e}^{-\left(\frac{\omega_{+}R_{١}}{\mathcal{D}}\right)^{٢}} \qquad (٣٣)$$

Using $R_{١} \ll ١$, Eq. (٣٣) reads

$$\frac{\omega_{+}}{٤\mu} - \frac{٤\alpha\sqrt{m_{ext}}}{\mathcal{D}} = ٠ \qquad (٣٤)$$

and, then, with the condition $R_{١} \ll ١$, oscillator frequency $\omega_{+} = \frac{٨\mu\alpha\sqrt{m_{ext}}}{(١+|\mathfrak{W}|)}$ of biexciton is defined and then this formula will help us to calculate the reduced mass μ of the biexciton system based on Eq. (٢٥). We calculate the result for the ground states of energy levels and oscillator frequency, respectively. As a result, Eqs. (٣٢) and (٣٣)) are described in the form

$$E = \frac{\omega_{+}^{٢}}{٨\mu} - \frac{٤\alpha\sqrt{m_{ext}}}{\mathcal{D}}\omega_{+}\;\frac{٤\alpha R_{١}^{٢}\sqrt{m_{ext}}}{\mathcal{D}^{٣}}\omega_{+}^{٣} \qquad (٣٥)$$

and

$$\omega_+^{٢} + \frac{\mathcal{D}^{٣}}{١٢\mu R_{١}^{٢}\alpha\sqrt{m_{ext}}}\omega_+ - \frac{٣\mathcal{D}^{٢}}{٣R_{١}^{٢}} = ٠ \qquad (٣٦)$$

Also, we determine the energy spectrum of the relativistic correction based on Eq. (٢٧) is defined by the following expression ($\mathcal{D} = ٢(١ + |\mathfrak{W}|)$)

$$\varepsilon_{٠rel} = -٤\alpha\,\mu\omega_+^{١+|\mathfrak{W}|}\int_{٠}^{\infty} du \frac{u^{|\mathfrak{W}|}e^{-u\omega_+}}{\Gamma(١+|\mathfrak{W}|)\sqrt{١+\frac{|\mathfrak{W}|(|\mathfrak{W}|+١)}{٤u^{٢}}}} \qquad (٣٧)$$

Since we know that the relativistic corrections in the absence of a strong external electromagnetic field can be neglected, therefore, by modifying Eq.(٢٢) based on the theoretical calculation from Eq. (١٨), we can determine the relativistic mass correction of an electron-hole cluster inside the biexciton system, and the resulting equation is solved as an equation $\mu_{ext} = \left(m_{ext}^{٢} - ٢\mu^{٢}\frac{dE}{d\mu}\right)^{١/٢}$, the reduced mass μ of the biexciton system is defined using the relation $\frac{١}{\mu} = \frac{٢}{\mu_{ext}}$. The results are calculated within the semiconductor quantum dot parameters for the $GaAs$, and the effective mass of electron and hole are taken as $m_e^* = ٠.٠٦٧m_e, m_h^* = ٠.٤٥m_e$ with the relative permittivity $\varepsilon_r = ١٣,١$. We consider specific parameters for $GaAs$ quantum dots due to tuning the strong electrostatic coupling constant. The coupling constant depends on the size, shape, charge distribution, and dielectric environment of $GaAs$ quantum dots and potential confinement. In this research, we consider the distance between electrons $R_١$: (٠.١-٥) a_B, where a_B is the Bohr radius. It is a characteristic length scale between electrons and in a biexciton system. The biexciton properties with the relativistic corrections on the total Hamiltonian, mass, and energy eigenvalue are calculated and presented in Table ١, Fig. ٣, and Fig. ٤.

TABLE I

BIEXCITON TO ELECTRON MASS RATIO IN THE STRONGLY OBLATE ELLIPSOIDAL $GaAs$ QUANTUM DOTS WITH COUPLING CONSTANTS (٥٠, ٨٠, ١٠٠) TIMES THE COULOMB INTERACTION (α_s).

	$٠.١a_B$	$٠.٥a_B$	$١a_B$	$٢a_B$	$٣a_B$	$٥a_B$
$M/m_{e_{(١٠٠\alpha_s)}}$	١,٧٢٨٧	١,٧٣١٧	١,٧٣٥٠	١,٧٣٩١	١,٧٤٨٠	١,٧٦٤٠
$M/m_{e_{(٨٠\alpha_s)}}$	١,٧٥٤٧	١,٧٥٥٩	١,٧٥٧٣	١,٧٥٩٢	١,٧٦٣٥	١,٧٧٢٤
$M/m_{e_{(٥٠\alpha_s)}}$	١,٧٨٢٤	١,٧٨٢٦	١,٧٨٢٨	١,٧٨٣١	١,٧٨٤٠	١,٧٨٦٠

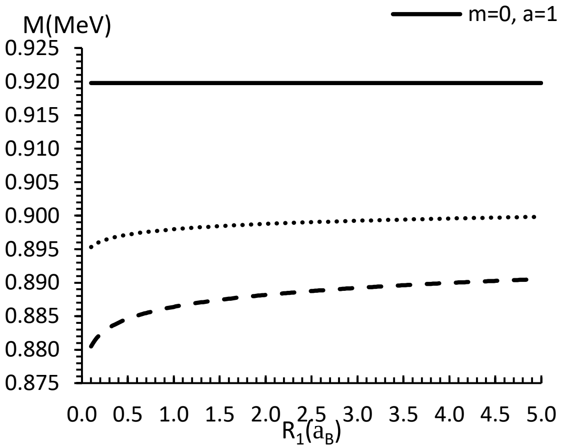

Fig. ٣. Biexciton mass as a function of the distance between electrons in SOEQD under coupling constant (١, ٨٠, ١٠٠) α_s at the ground state.

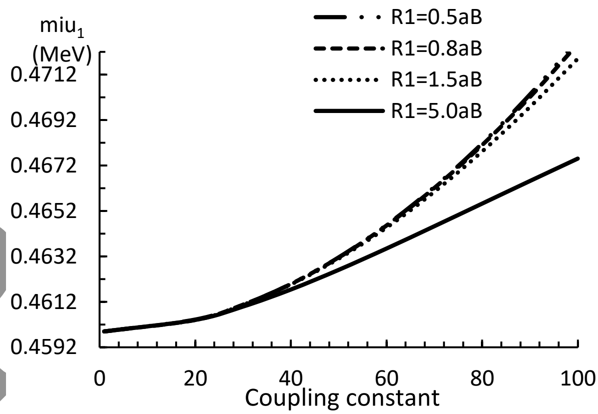

Fig. ٤. The constituent mass of exciton in the biexciton system as a function of coupling constant (١-١٠٠) α_s in SOEQD with different electrons distant R.

٦. CONCLUSION

Biexciton's energy levels in the strongly oblate ellipsoidal *GaAs* semiconductor quantum dots have interesting optical properties and characteristics due to strong interaction near their confinement. Biexciton's energy level has a different value from the single exciton clusters inside a bound system and it is typically larger than a single exciton in a bound state. Hence transition energy of biexciton is higher when a biexciton-bound state is created or annihilated. Biexciton based on the energy levels absorbs and emits optical light at specific energy eigenvalues. As we know, the absorption spectra of biexcitons are directly connected to the energy eigenvalues of the system, which allow all energy levels that a biexciton in the strongly oblate ellipsoidal *GaAs* semiconductor quantum dots can possess. In strong Coulombic interaction near the confinement of particles, biexcitons can interact with other particles, photons, phonons, and excitons in *GaAs* semiconductors. These interactions directly affect the quantum-optical and optoelectrical properties and lead to a modification of the quantum-optical properties, binding energy, radiative recombination rate, linewidth, lifetime, spectral shape of photoluminescence of biexciton compared to single excitons, which can describe by energy eigenvalues. Hence, the description of energy eigenvalues based on the relativistic correction can be useful to understand new characteristics and behaviors of biexciton systems. As it is widely recognized biexcitons in *GaAs* in the strongly oblate ellipsoidal nanoscale structures with Coulombic interaction and strong quantum confinement, have exhibit size-dependent optical properties. Hence, The transition energies of biexcitons can be tuned by controlling the dot size and shape. These quantum-optical properties of biexcitons in *GaAs* strongly oblate ellipsoidal form is the subject of extensive theoretical and experimental investigations, as they have consequences in hi-tech devices, quantum information processing, nanoelectronics technologies, and fundamental studies of solid state and semiconductor physics. In summary, we can present that some of the biexciton state properties in the SPEQD were calculated theoretically in this article showing that the results can be compared with the variational method described in [۳۲, ۳۳].

The physical concepts included in this article are as follows:

- The relativistic effect and the geometry of the quantum dot play important roles in the formation of the spectrum of charge carriers that constitute the exciton or biexciton systems.
- Relativistic corrections on the electron-hole bound states can manifest as specific properties in quantum dots, leading to the introduction of new corrections on the physical quantities of the system.
- The relativistic correction presented in this article can affect the characteristics of excitons, biexcitons, negative/positive biexcitonic ions, and trions. It can influence nonlinear susceptibility, two-photon absorption, the appearance of new resonance lines, and the shift of photoluminescence emission peaks.

- Since biexcitons are bound states of electron-hole pairs, the inclusion of relativistic corrections allows us to describe new trends in the theory of biexciton-polariton interaction, the electronic structure of biexcitons, and the radiative decay of biexcitons.

REFERENCES

[۱] D.B. Hayrapetyan, *Binding and recombination energies of quasi-one-dimensional excitonic complexes in ellipsoidal quantum dot*, Foundations,. ۲(۱), ۲۱۹-۲۲۷, ۲۰۲۲. https://doi.org/۱۰,۳۳۹۰/foundations۲۰۱۰۰۱۵;

[۲] *S. damizadeh, et al, Electronic and Optical Properties of SnGe and SnC Nanoribbons: A First-Principles Study,* Journal of Optoelectronical Nanostructures, ٥(٤), ٦٧-٨٦, ۲۰۲۰. ۲۰,۱۰۰۱,۱,۲٤۲۳۷۳٦۱,۲۰۲۰,٥,٤,٥,٦

[۳] H.H. Xie, L.G. Jiao, A. Liu, and Y.K. Ho, *High-precision calculation of relativistic corrections for hydrogen-like atoms with screened Coulomb potentials*, International Journal of Quantum Chemistry, ۱۲۱(۱۳) e۲٦٦٥۳, ۲۰۲۱. https://doi.org/۱۰,۲۲٥٤۱/au.۱٦۱۱۷٤۸۰۹,۹۱۳۲۲٦۸۹/v۱.

[٤] F. Rahimi, et al., *Study the Energy States and Absorption Coefficients ofQuantum Dots and Quantum Anti-Dots with Hydrogenic Impurity Under the Applied Magnetic Field*, Journal of Optoelectronical Nanostructures, ۷(۱), ۱-۸, ۲۰۲۲. https://doi.org/۱۰,۳۰٤۹٥/jopn.۲۰۲۱,۲۸۷۸٤,۱۲۳۲

[٥] K.G. Dvoyan, D.H., EM Kazaryan, *Direct interband light absorption in a strongly oblated ellipsoidal quantum dot*, Nanoscale research letters,. ٤, ۱۰٦-۱۱۲, ۲۰۰۹. https://doi.org/۱۰,۱۰۰۷/s۱۱٦۷۱-۰۰۸-۹۲۰۹-۲.

[٦] M.Hsani, Raad Chegell, *Electronic and optical properties of the graphene and boron nitride nanoribbons in presence of the electric field,* Journal of Optoelectronical Nanostructures, ٥(۲), ٤۹-٦٤, ۲۰۲۰. ۲۰,۱۰۰۱,۱,۲٤۲۳۷۳٦۱,۲۰۲۰,٥,۲,٥,۲

[۷] D. Hayrapetyan, *Direct interband light absorption in a strongly prolated ellipsoidal quantum dot*, Journal of Contemporary Physics (Armenian Academy of Sciences), ٤۲,. ۲۹۲-۲۹۷, ۲۰۰۷. https://doi.org/۱۰,۳۱۰۳/S۱۰٦۸۳۳۷۲۰۷۰٦۰۱۷٥.

[٨] M. Servatkhah, P. Hashemi, R. Pourmand, *Binding energy in tuned quantum dots under an external magnetic field*, Journal of Optoelectronical Nanostructures, ٧(٤), ٢٠٢٢. https://doi.org/١٠,٣٠٤٩٥/jopn.٢٠٢٢.٣٠٩٢٤,١٢٧٠

[٩] D.A. Baghdasaryan, D.B. Hayrapetyan, and E.M. Kazaryan, *Optical properties of narrow-band prolate ellipsoidal quantum layers ensemble*, Journal of Nanophotonics, ١٠(٣) ٠٣٣٥٠٨, ٢٠١٦. https://doi.org/١٠,١١١٧/١.JNP.١٠,٠٣٣٥٠٨

[١٠]Y.Y. Bleyan, et al., *Non-linear optical properties of biexciton in ellipsoidal quantum dot,* Nanomaterials, ١٢(٩), ١٤١٢, ٢٠٢٢. https://doi.org/١٠,٣٣٩٠/nano١٢٠٩١٤١٢

[١١]C. Chang, X. Li, X. Wang, and C. Zhang, *Nonlinear optical properties in GaAs/Ga٠. ٧Al٠. ٣As spherical quantum dots with Like-Deng-Fan-Eckart potential*, Physics Letters A, ٤٦٧, ١٢٨٧٣٢, ٢٠٢٣. https://doi.org/١٠,١٠١٦/j.physleta.٢٠٢٣,١٢٨٧٣٢

[١٢]N. Moshayedi, *Construction of Quantum Field Theories, in Quantum Field Theory and Functional Integrals: An Introduction to Feynman Path Integrals and the Foundations of Axiomatic Field Theory*, Springer, ٢٠٢٣.

[١٣]V. Kumar, S. Bhardwaj, R.M. Singh, and F. Chand, *Optical properties and effect of magnetic field on energy spectra of a GaAs spherical quantum dot*, The European Physical Journal Plus, ١٣٨(٣), ١٩١, ٢٠٢٣. https://doi.org/١٠,١١٤٠/epjp/s١٣٣٦٠-٠٢٣-٠٣٨١٠-y

[١٤]Z.-H. Zhang, and J.-H. Yuan, *Electronic and nonlinear optical properties in AlxGa١− xAs/GaAs Gaussian confinement quantum dot under applied electric field*, Physica E: Low-dimensional Systems and Nanostructures, ١٤٧, ١١٥٥٩٤, ٢٠٢٣. https://doi.org/١٠,١٠١٦/j.physe.٢٠٢٢,١١٥٥٩٤; S. Damizadeh, M. Nayeri, F. Kalantari Fotooh, and S. Fotoohi, *Electronic and Optical Properties of SnGe and SnC Nanoribbons: A First-Principles Study*, Journal of Optoelectronical Nanostructures, ٥(٤) ٦٧-٨٦, ٢٠٢٠. https://doi.org/١٠,٣٠٤٩٥/jopn.٢٠٢٣,٣١٣٣٢,١٢٧٧

[١٥]M. J. Ambrosio, et al., *Mathematical properties of generalized Sturmian functions,* Journal of Physics A: Mathematical and Theoretical, ٤٥(١), ٠١٥٢٠١, ٢٠١١. https://doi.org/١٠,١٠٨٨/١٧٥١-٨١١٣/٤٥/١/٠١٥٢٠١

[١٦]G. Gasaneo, et al., *Three-Body Coulomb problems with generalized Sturmian functions*, Advances in Quantum Chemistry, ٦٧, ١٥٣-٢١٦, ٢٠١٣. https://doi.org/١٠,١٠١٦/B٩٧٨-٠-١٢-٤١١٥٤٤-٦,٠٠٠٠٧-٨

[١٧]M. Dineykhan, G. Efimov, G.d. Ganbold, and S. Nedelko, *Oscillator representation in quantum physics*, Springer Science & Business Media, ٢٠٠٨, ١٢٠-١٩٨

[١٨]A. Jahanshir, *Quanto-relativistic background of strong electron-electron interactions in quantum dots under magnetic field*, Journal of Optoelectronical

Nanostructures, ٦(٣), ١-٢٤, ٢٠٢١. https://doi.org/١٠,٣٠٤٩٥/JOPN.٢٠٢١,٢٨٧٤٢,١٢٣١

[١٩]E.Hiyama, Y. Kino, and M. Kamimura, *Gaussian expansion method for few-body systems*, Progress in Particle and Nuclear Physics, ٥١(١), ٢٢٣-٣٠٧, ٢٠٠٣. https://doi.org/١٠,١٠١٦/S٠١٤٦-٦٤١٠(٠٣)٩٠٠١٥-٩

[٢٠]M.A Gosson, *Symplectic geometry and quantum mechanics,* Springer Science & Business Media, ٢٠٠٦, ٦٥-٩٤.

[٢١]M. Bojowald, *Canonical description of quantum dynamics*, Journal of Physics A: Mathematical and Theoretical, ٥٥(٥٠), ٥٠٤٠٠٦, ٢٠٢٣. https://doi.org/١٠,١٠٨٨/١٧٥١-٨١٢١/acafb٠

[٢٢]R.P. Feynman, A.R. Hibbs, and D.F. Styer, *Quantum mechanics and path integrals*, Courier Corporation, ٢٠١٠, ٧٣-١٥٠.

[٢٣]M.E. Peskin, *An introduction to quantum field theory*, CRC press, ٢٠١٨.

[٢٤]R. Rosenfelder, *Path integrals in quantum physics,* arXiv preprint arXiv:١٢٠٩,١٣١٥, ٢٠١٢. https://doi.org/١٠,٤٨٥٥٠/arXiv.١٢٠٩,١٣١٥

[٢٥]A. Benkrane, *Position dependent mass and the extended uncertainty principle of quantum dynamics*, Eliva Press SRL, ٢٠٢٣, ٤٠-٨٦.

[٢٦]M. Douici, H. Boukabcha, and R. Fermous, *Study of energies spectra and thermodynamic properties of the relativistic Dirac equation using Feynman path integral method*, Physica Scripta, ٩٨, ٠٧٥٤٠٥, ٢٠٢٣. https://doi.org/١٠,١٠٨٨/١٤٠٢-٤٨٩٦/acdc٦٠

[٢٧]M. Dineykhan, G. Efimov, and K. Namsrai, *Investigation of Green Functions and the Parisi-Wu Quantization Method in Background Stochastic Fields*, Fortschritte der Physik/Progress of Physics, ٣٩(٤), ٢٥٩-٣١٨, ١٩٩١. https://doi.org/١٠,١٠٠٢/prop.٢١٩٠٣٩٠٤٠٢

[٢٨]A. Jahanshir, E. Omugbe, *Relativistic mass of spherical layered quantum dots within harmonic potential*,٢nd *National Conference on Technological Advances in Applied Physics*, Graduate University of Advanced Technology, Kerman, Iran, ٢٠٢٣. https://civilica.com/l/١٠٣٠٨٣/

[٢٩]M.I. Eides, H. Grotch, and V.A. Shelyuto, *Theory of light hydrogenlike atoms*, Physics Reports, ٣٤٢(٢-٣), ٦٣-٢٦١, ٢٠٠١. https://doi.org/١٠,١٠١٦/S٠٣٧٠-١٥٧٣٪٢٨٠٠٪٢٩٠٠٠٧٧-٦

[٣٠]H. Yukawa, *On the interaction of elementary particles*, Proceedings of the Physico-Mathematical Society of Japan. ٣rd Series, ١٧, ٤٨-٥٧, ١٩٣٥. https://doi.org/١٠,١١٤٣/PTPS.١,١

[٣١]M. Dineykhan, and R. Nazmitdinov, *Ground state spin oscillations of a two-electron quantum dot in a magnetic field*, Journal of Physics: Condensed Matter,. ١١(١١) L٨٣, ١٩٩٩. https://doi.org/١٠,١٠٨٨/٠٩٥٣-٨٩٨٤/١١/١١/٠٠٢

[٣٢] D. Hayrapetyan. *Binding and recombination energies of quasi-one-dimensional excitonic complexes in ellipsoidal quantum dot*, Foundations journal,

۲(۱) ۲۱۹–۲۲۲, ۲۰۲۲. https://doi.org/۱۰,۳۳۹۰/foundations۲۰۱۰۰۱۵
[۳۳]L. Bao, et al, *On the optical stark effect of excitons in InGaAs prolate ellipsoidal quantum dots*, Journal of Nanomaterials, ۵۵۸٦٦۲۲, ۱-۱۲, ۲۰۲۱. https://doi.org/۱۰,۱۱۵۵/۲۰۲۱/۵۵۸٦٦۲۲