Islamic Azad University

Journal of

Optoelectronical Nanostructures

Summer 2021 / Vol. 6, No. 3

Research Paper

Quanto-Relativistic Background of Strong Electron-Electron Interactions in Quantum Dots under the Magnetic Field

Arezu Jahanshir[*,1]

[1] Department of Physics and Engineering Sciences, Imam Khomeini International University, Buein Zahra Higher Education Centre of Engineering and Technology, Iran

Received: **4 Jun 2021**
Revised: **16 Jul. 2021**
Accepted: **15 Aug. 2021**
Published: **15 Sep. 2021**

Use your device to scan and read the article online

Keywords:
Energy Eigenvalues, Gaussian Asymptomatic, Relativistic Effective Mass, Schrödinger Equation

Abstract

At a finite temperature, electron-electron interactions and energy eigenvalues were studied using in the field of symplectic geometry and the relativistic radial Schrödinger equation with the expanded exponential thermal potential (parabolic potential) representing the strong electron-electron interaction. Electron-electron interactions can strongly affect the effective mass, mass spectrum, and functionality of multi-electron quantum dots. A quanto-relativistic interaction's behavior and effects with temperature dependence in the magnetic field are shown to have a unique feature in semiconductor quantum dots. These results have important implications for lighting, quantum dot enhancement film, rational design, edge optic, new materials, spin electronics color filter, on-chip, visible and IR/NIR image sensor, photovoltaic, and fabrication of quantum dot qubits with predictable properties.

Citation: Arezu Jahanshir. Quanto-Relativistic Background of Strong Electron-Electron Interactions in Quantum Dots Under the Magnetic Field.
Journal of Optoelectronical Nanostructures, 2021; 6 (3): 1- 24.
DOI: 10.30495/JOPN.2021.28742.1231

***Corresponding author:** Arezu Jahanshir
Address: Department of Physics and Engineering Sciences, Imam Khomeini International University, Buein Zahra Higher Education Centre of Engineering and Technology, Iran. **Tell:** 00982833894492 **Email:** jahanshir@bzeng.ikiu.ac.ir

1. INTRODUCTION

It is essential to investigate the electron-electron interaction's finite temperature dependence at great energy and intense contact. This connection holds for various possible models and explains a previously electron-electron interactions characteristic. In quantum dots and a very small volume and a slice of semiconductor, when few electrons are closely and firmly confined, they tend to occupy well-defined positions giving rise to the so-called "Wigner molecule", a strongly correlated electron state of the matter, especially the solid and liquid phases of semiconductor quantum dots which arise from electromagnetic forces between particles. It is named after "the strong electron-electron interactions" and is characterized by strong fluctuations of the correlation function. We utilize quantum field techniques based on the creation and annihilation operators to tackle the study of this exotic and strong interaction system. The main study of this topic is the finite temperature effect and relativistic mass corrections on the strong electron-electron interaction in the magnetic field. This specific strong interaction occurs in quantum dots with low electron densities, which are analogs of Wigner molecules. Experimental signatures of Wigner molecules include the suppression of multi-electron excitation energies, which has been observed in SQDs [1]. In this paper, we focus on the relativistic mass effects based on the energy-momentum relation $E^2=(pc)^2+(m_0c^2)^2$, the Schrödinger equation in which is equivalent to the behavior of a system in the strong electron-electron interaction in the quantum dots. Based on the Power series expansion in the Schrödinger equation and exponential thermal potential, the behavior of a system is presented. Relativistic effects are taken into account by expanding $\sqrt{p^2+m^2}$ in a lowest order power series of momentum and mass under the equation $\left[\sum_{i=1}^{n}\sqrt{p_i^2+m_i^2}+U(r)\right]\Psi(r)=E\Psi(r)$. The radial Schrödinger equation, and an analytic approach based on the symplectic geometry method of strong interactions in a finite temperature electrostatic field, are used to depict interaction in temperature-dependent terms [1-3]. The binding energy and effective mass of the strong electron-electron interactions are calculated using this approach. Strong electron-electron interactions and exotic molecular states do not conform to more conventional states. They include states involving electron-electron and electron-hole states. The exotic strong electron-electron interactions are multibody (charged) states investigated using exponential potential and framework techniques, including the molecular model, atomic model, the multiparticle coupling model, the two-electron quantum dot Hamiltonian, the tight-binding model, and full configuration method. Recent experimental studies of strong quanto-relativistic interactions have revealed many states: electron-electron,

electron-light hole, electron-heavy hole, di-exciton, quadro-exciton, etc. [4, 5]. At finite temperatures, it has been demonstrated that the effective masses of electrons and holes can be significantly larger than anticipated theoretically. The Los Alamos National Laboratory Collaboration, LANL's National High Magnetic Field Lab, and the spectroscopy capabilities in the LANL chemistry labs are projected to take a new dimension and open up new study areas for semiconductor quantum dots at finite temperature in a brand-new operation [6]. Thus, theoretical investigations can raise awareness and intrigue in experimental interpretations. Strong electron-electron interactions in the form of exotic molecules are investigated in this research. The energy and effective mass of a multi-electron system with exponential potential in the ground states are determined using the correlation functions' Gaussian asymptomatic behavior. Furthermore, a relativistic correction to the component mass is derived. The Schrödinger equation and the constituent electron component's mass calculate the exotic molecule's (constituent electron-electron interactions) energy eigenvalue and effective mass. The exotic molecule mass component is defined via modifying correction. The effects can be localized, as in the case of exotic molecules (when few electrons are tightly confined in a small slice of matter, they tend to occupy well-defined positions, giving rise to the so-called exotic molecules, a strongly correlated electron state of the quantum dots), with consequences for the energy spectrum to states outside the dots. Multi-particle models are effective in describing strong electron-electron interactions and exotic molecular state mass. Consequently, the strong electron-electron interactions are investigated concerning the electrons constituent system at a finite temperature in the external magnetic field $B(x)$; this notion, in conjunction with the radial Schrödinger equation, determines the strong quanto-relativistic characteristic of semiconductor quantum dots [7,8].

A. Research Aim

In this study, the strong electron-electron interactions in semiconductor quantum dots are defined in this theoretical research as the Schrödinger equation solution of a quanto-relativistic background in the exponential potential at finite temperature. This opinion was formed using PUR and technique [9-11]. As is apparent, the electrons' behavior in their strong interactions near their confinement temperature is essential in settings with strong electron-electron interactions, including two, three, and multi electrons semiconductor quantum dots. Calculations using different quanto-relativistic models often provide imprecise results, and we cannot forecast the relativistic effective mass value. The electrons' effective mass and energy eigenvalues must be predicted using quanto-

relativistic behavior. Consequently, the energy eigenvalues at a given temperature are presented using quantum field theory, and a connection between the relativistic effective mass with temperature is established.

B. Projective unitary

The behavior of the mass spectrum, the bound states, and the relativistic corrections to the Hamiltonian and mass in a strongly interacting environment at the finite temperature or in the confining potential is very important, especially in nano quantum dots and nanomaterials. The various model predicts the mass with temperature variation based on different solving methods by the Schrödinger equation (SE) or other equations. In a theoretical and analytical manner, the projective unitary technique is one of the effective methods for solving the SE. Therefore, the inclusion of the relativistic bound state in quantum dots at finite temperature is one of the most interesting subjects of contemporary theoretical physics. We can formulate the Hamiltonian by the PUR method for the quantum system in the R^d space, in this case, the Hamiltonian $H = \sum_{i=1}^{d} \frac{1}{2}\left(\hat{p}_i^2 + \omega^2 \hat{q}_i^2\right)$ (in natural units $\hbar = c = 1$) using the canonical variables $(\hat{q}, \hat{p})$ in the form of creation $\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{q} - \frac{i}{\sqrt{\omega}}\hat{p}\right)$ and annihilation $\hat{a}^- = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{q} + \frac{i}{\sqrt{\omega}}\hat{p}\right)$ operators can use to find the eigenvalues of these Hermitian operators, where $[\hat{a}^-, \hat{a}^+] = d$, $\hat{q}$ and $\hat{p}$ are the canonical operators for the particle and satisfy $[\hat{q}, \hat{p}] = i$ In other words, the PUR completely describes and presents the Hamiltonian in the correct form of these operators [12]. The modified Schrödinger equation $H\Psi(q) = \varepsilon(E)\Psi(q)$ in accordance with creation and annihilation operator in PUR, we determine that the energy spectrum in d- demission space is $\varepsilon(E) = 0$ and $\frac{\partial \varepsilon(E)}{\partial \omega} = 0$ these conditions are the main technique for determining the eigenenergy of Hamiltonian [12].

2. MATERIALS AND METHODS

Using the analytic method based on the behavior of the correlation function of the related field flows of charged particles is suggested [12] for determining the relativistic effect of strong electron-electron interactions in quantum dots at a finite temperature. This idea exactly can determine the quanto-relativistic correction to the effective mass of particles. Therefore, the component effective mass of the electron is defined by modifying corrections to the radial Schrödinger equation $\widehat{H}\Psi(r) = E\ (\mu)\Psi(r)$in the new axillary coordinate, and then the quanto-relativistic characteristic is determined [12]. This is a good approximation for

defining the relativistic properties of quantum dots in strong and weak interactions. We determined the corrections to the effective mass of electrons in the confining potential, based on PUR. The modified radial Schrödinger equation (MRSE) obtains the strong electron-electron interactions Hamiltonian in the framework of quantum field theory and the scattering matrix using the appropriate Feynman diagram by considering the renormalization and the non-relativistic limit. The techniques and manners of quantum field theory have been proposed to present the binding energy and relativistic behavior of different systems with fairly arbitrary potentials presented by the SE. The relativistic mass and the effective mass are determined by the correlation function $\Pi(r) = \left\langle G_{m_1}(r|A) G^*_{m_2}(r|A) \right\rangle_A$ of the corresponding current of the field with the quantum numbers. This is presented in terms of the Green's function and the Feynman's functional path integral in non-relativistic quantum mechanics [11, 12]. Therefore, the correction to the mass is defined as a limit of the correlation function in the asymptotic limit $M = -\lim_{|r|\to\infty} \frac{ln\,\Pi(r)}{|r|}$. Therefore, the strong particles interactions Hamiltonian in the framework of quantum field theory (QFT) with correlations can be realized as a bound state if $M \neq m_1 + m_2 + \cdots, M < \infty$, then a binding condition (bound state) with a mass M arises; if $M = m_1 + m_2 + \cdots$, then the effective interaction cannot arise a stable and clear bound state and the scalar particles exist as two quanto-relativistic independent states with relativistic effective masses in quantum dots [12]. As we know, the radial SE for the relative coordinate of the system and the center of mass has been described based on the possibility of strong electron-electron interactions in quantum dots by

$$\left(\sum_{i=1.\ i\neq j}^{n} \frac{1}{2\mu_e^*}\left(P_i + \frac{e}{c}A_i\right)^2 + U_{total} + H_{spin}\right)\Psi = E(\mu)\,\Psi \qquad (1)$$

So, quantum field theory describes systems with finite temperature as a limitless number of oscillators maintaining their oscillating characteristics throughout interactions. According to the PUR model and symplectic geometry method for the wave function $\Psi(r)$, we have to represent variables to describe a harmonic oscillator behavior for the wave function of the converted and transformed modified SE equation and then describe the radial Schrödinger equation in a new space with a different dimension [13,14]. The wave function must decrease at small distances, so the transformation to the upper dimensional space is realized by $r \to q^2$. To use quantum field techniques, [12, 15] must be replaced new variables in the MRSE:

$$\widehat{H}\,\Psi(r)=E\,(\mu)\;\Psi(r)\rightarrow\Psi\,(r\,)\sim q^{2\rho}\;\Psi\big(q^2\big)=q^2\;\Psi\big(q^2\big)\,. \tag{2}$$

When more considerable distances are involved, the wave function must have a Gaussian solution; therefore, the Hamiltonian is used with the PUR variables from the equation. In a new auxiliary space, the radial part of the SE for the relative motion of two-electron QDs with strong electron-electron interactions at a finite temperature in three dimensions using the radial Laplacian

$$\Delta_r=\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\rightarrow\Delta_q=\frac{d^2}{dq^2}+\frac{\mathcal{D}-1}{q}\frac{d}{dq}, \tag{3}$$

is defined. Here $\mathcal{D}=4+4\ell$, and now under this transformation, Eq. (1) is presented as follows ($\varepsilon_0=\hbar=c=1$)

$$\left[\frac{d^2}{dq^2}+\frac{4\ell+3}{q}\frac{d}{dq}+4\mu^*q^2\,(U_{total}-E(\mu)\,)\right]\Psi(q^2)=0\rightarrow$$

$$\left[\frac{P_q^2}{2}+4\mu^*q^2\,(U_{total}-E(r)\,)\right]\Psi(q^2)=0, \tag{4}$$

$$\hat{q}=\frac{1}{\sqrt{2\omega}}(\hat{a}^-+\hat{a}^+),\;\hat{p}_q=-i\sqrt{\frac{\omega}{2}}(\hat{a}^--\hat{a}^+),$$

where $\hat{a}^+,\hat{a}^-$ denote the creation and annihilation operators, respectively. Based on the PUR condition, the corresponding canonical variables are derived as Wick orderings [14]:

$$P_q^2=2\omega\,(1+\ell).\quad q^2=\frac{2}{\omega}(1+\ell).\qquad q^4=\frac{2}{\omega^2}(1+\ell)(3+2\ell), \tag{5}$$

The interaction Hamiltonian includes all non-square components (a condition in Wick ordering), so we can then discover the renormalization of the bound state parameters, including the wave function, which enables us to introduce the PUR using the zero approximation and afterward find the eigenvalue of the ground state energy $\varepsilon_0(E,T)$. Therefore, the following Eq. (4) is constructed based on Eq. (5) (for a more thorough explanation, see [12]):

$$\varepsilon_0(E\,,T)=\frac{d\omega}{4}-4\mu^*q^2\,(U_{total}-E(\mu)\,)=0. \tag{6}$$

The following Eq. (6) may therefore be rewritten:

$$\varepsilon_0(E.T) = X(T.\omega) - EY(T.\omega) = 0.$$

And then, using PUR conditions, we determine

$$\varepsilon_0(E,T) = 0, \frac{d\varepsilon_0(E,T)}{d\omega} = 0. \quad (7)$$

The pure oscillator frequency ω_0 and the energy eigenvalue $E(r)$ of strong electron-electron interactions in quantum dots may now be determined and using zero approximation from Eq. (7) [12]. Thus, using the radial Schrödinger equation $\widehat{H}\Psi(r) = E(\mu)\Psi(r)$ and the steepest descent point, the masses of the particles in the strong interactions are determined as follows ($i = 1,2$):

$$M = \frac{\partial}{\partial\mu_i}\left(\left(\frac{\mu_2 m_1^2 + \mu_1 m_2^2}{2\mu_1\mu_2}\right) + \frac{\mu}{2} + E(\mu)\right) \rightarrow M = \mu_1 + \mu_2 + \mu E'(\mu) + E(\mu). \quad (8)$$

$$\mu_i = \frac{\partial M}{\partial\mu_i} = 0 \Rightarrow \mu_1 = \sqrt{m_1^2 - 2\mu^2 E'(\mu)}. \quad \mu_2 = \sqrt{m_2^2 - 2\mu^2 E'(\mu)}. \quad (9)$$

where $E'(\mu) = \frac{\partial E(\mu)}{\partial\mu}$, and $\mu = \frac{\mu_1+\mu_2}{\mu_1\mu_2}$ is the reduced mass of the system, μ_i is the constituent mass of particles, and m_i is the rest mass of particles. In the next paragraph within the suggested technique, the analytical terms for the quanto-relativistic effect on the strong electron-electron interactions in quantum dots at a finite temperature are determined.

3. PARABOLIC-EXPONENTIAL POTENTIAL

The studies propose that the charged component particles of electron-electron interactions at finite temperature, particularly electrons, may also be utilized to define the new property of effective masses, providing a plausible approximation for describing the features of quantum dots in both strong and weak interactions at finite temperature. The electron-electron interactions in a confining potential at a finite temperature (T) for quantum dots using the exponential $U(r,T)$ and Coulomb $U(r)$ potential models are calculated in this paper based on the exponential potential:

$$U(r,T) = V_0\left(1 - e^{-(a_T r)^2}\right), \quad (10)$$

where parameter V_0 is the depth of the region surrounding a local minimum of potential energy i.e. the depth of the potential well. a_T is the parameter that describes and includes temperature relation and the range of the confinement potential (CP) and its properties such as frequency. In the very small QDs, a good approximation of interaction is a Gaussian type potential when the dimension is of the same order as the characteristic length of the variation of CP. Therefore, we choose the confining potential to be the spherically symmetric Gaussian potential well form:
$U(r) = -Be^{-\beta^2 r^2}$, $(\beta^2 > 0, B > 0, -\infty \ll r \leq +\infty)$, due to facilitating comparison of the model with the harmonic potential type and the wave function of SE for this type of potential well can present in the form $\Psi(x) = (\frac{\xi}{\pi})^{d/4} e^{-\xi^2 r^2}$. Hence, we can set it $B \equiv \frac{\mu_e^* \omega_0^2}{2\beta^2}$ for the Gaussian potential well. In some other studies, we can set the depth of the potential well $B=\gamma R_D$, where R_D the Rydberg and is the unit of energy, γ is constant value depends on the material parameters and the certain amount number of electrons in the system (one-electron QDs for the GaAs quantum dots embedded in ($Al_{0.3}$ Ga $_{0.7}$ As) matrix, $B=50R_D$ [16, 17]). As an exponential function represents the nonzero temperature parameter (and other properties of potential) following the parameter $a(T) = a_T$, the given function may be changed as follows using

$$e^{-ax} = \sum_{n=0}^{\infty} \frac{(-ax)^n}{n!} = 1 - ax + \frac{1}{2}(ax)^2 + \cdots,$$

then the exponential approximation limits of (10) read:

$$e^{-(a_T r)^2} = \sum_{n=0}^{\infty} \frac{(-(a_T r)^2)^n}{n!} = 1 - (a_T r)^2 + \frac{1}{2}(a_T r)^4 + \cdots. \quad (11)$$

Therefore, one can find

$$U(r,T) \cong V_0 \left((a_T r)^2 - \frac{1}{2}(a_T r)^4 + \cdots \right) \approx V_0 (a_T r)^2, \quad (12)$$

where $a(T) = 0$ at absolute zero temperature. In order to solve RSE analytically, it is necessary to separate the electron-electron interaction Hamiltonian into two different parts, one corresponding to the center of mass and the other to the relative motion of the two-electron system. That is the main reason and motivation to present and estimate the confining exponential potential to be harmonic type potential model for which the total Hamiltonian is separable in the two different

semi-coordinates because the Hamiltonian is not separable for the Gaussian type potential, in this case with some mathematical presentation one can divide the Hamiltonian that should provide an estimated model for realistic quantum dots. To simplify the comparison of the presentation in (12) with the parabolic potential model, we set $V_0 = \frac{\omega_0^2 \mu_e^*}{2\beta^2} = \frac{\omega_0^2 \mu_e^*}{2a_T^2}$, and then

$$U(r,T) \cong V_0 (a_T r)^2 \equiv \frac{1}{2}\omega_0^2 \mu_e^* (a_T r)^2 \rightarrow \omega_0^2 \equiv \frac{2a_T^2 V_0}{\mu_e^*} \quad . \tag{13}$$

Therefore, the radial Schrödinger equation has to be changed to solve and explain the strong electron-electron interactions at a finite temperature [16, 17]. Parameter $a(T)$ sets a scale for the screening of static charges and the scattering of temperature within a quantum dot. It is equivalent to the Debye mass in gauge plasma. The effective strong electron-electron interactions at a finite temperature of two electrons in QDs are determined according to the assumption that the charge images caused by the large difference between primitivity values of different layers are of considerable importance in the formalism of the electron confinement of QDs. The electron-electron interaction's effective potential was considered with spherically symmetric interactions. Hence, we have examined two-body interactions comprising two electrons; Let $m_e^*, -e$, are the nonrelativistic effective electron mass and charge of the electron.

The Hamiltonian of strong electron-electron interactions in PUR is described for two interacting electrons in QDs. Dineykhan and Efimov [12] developed the PUR technique from conception and properties of the quantum field theory. Using the PUR the characteristic of two-electron and multi-electron QDs in a magnetic field [18] have been calculated. The PUR results agree very well with the results obtained by variational numerical methods and analytic methods for these potentials. Now, we use the PUR technique to obtain an analytic solution for the strong electron-electron quantum dot confined by an exponential potential at a finite temperature in a magnetic field under the effective confining parabolic potential[19-20]. The Hamiltonian of strong electron-electron interaction is separated in the center of mass (H_c), and the relative (H_r) coordinate and spins (H_{spin}) with particles velocity (v_e) and it is defined as [12,21]:

$$\widehat{H}\Psi(r) = E(\mu_e^*)\,\Psi(r). \widehat{H} = H_c + H_r + H_{spin} \rightarrow$$

$$E\left(\mu_e^*\right) = E_{c(n_r.m_\ell)} + E_{r(n_r.m_\ell)} + E_{spin(s,m_\ell)} ,$$

$$\hat{H} = \sum_{i=1,i\neq j}^{n} \frac{1}{2\mu_e^*}\left(P_i + \frac{e}{c}A_i\right)^2 + \sum_{i=1.\ i\neq j}^{n} U_{ij}(r_{ij}) + U(r_{ij}.T) + H_{spin}, \qquad (14)$$

n is the number of particles in the QDs, and μ_e^* indicate the constituent's quanto-relativistic effective mass of particles in the strong electron-electron interaction states, distinct from the remaining particles effective mass (m_e^*) and particles mass at rest (m_e) in the electron-electron interactions in the quantum dots. Then

$$\hat{H} = \sum_{i=1.i\neq j}^{n} \frac{1}{2\mu_e^*}\left(p_i + \frac{e}{2\mu_e^*}\left[B(r_{ij}) \times r_{ij}\right]\right)^2 + V_0\, a_T^2 r_{ij}^2 + \frac{e^2}{4\pi\varepsilon_r\,\varepsilon_0 r_{ij}} + H_{spin}, \qquad (15)$$

where CP in quantum dots is characterized by a strong blocking along with one of the coordinate axes, and the external magnetic field can be directed along the perpendicular plane to the quantum dot, $\mu_B = \frac{\hbar e}{2cm_e}$ is the Bohr magneton, ε_r , ε_0 are the relative and absolute permittivity, H_{spin} is the Hamiltonian of the spin interactions, $r_{ij} = |r_i - r_j|$, $B(r_{ij})$ is the external magnetic field and it is oriented in a plane perpendicular to the plane of QDs [22,23], V_0 is the constant parameter of CP, $A(r) = 0.5[B(r) \times r]$ is the vector potential $(A(y) = 0)$ and low-lying quantum excitations are determined by the properties of the confinement potential along the remaining two axes, $\mu_e{}^*$ is the relativistic effective mass of electrons in strong electron-electron interactions under the magnetic field in the two electrons quantum dots and ω_B is the cyclotron frequency $B(r_i) = \frac{\omega_B \mu_e{}^*}{e}$. Now the Jacobi $(J_1.J_2)$ coordinates (where (J_2) is the center of mass of two electrons system) are introduced in Fig.1:

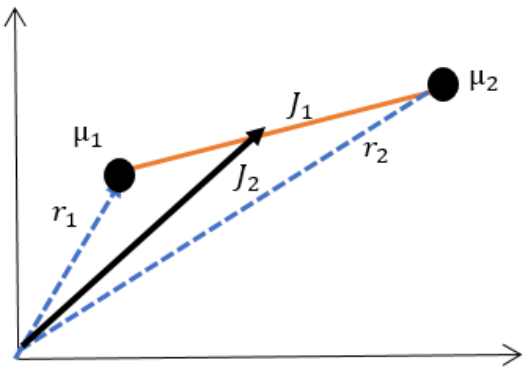

Fig. 1. The Jacobi coordinates of two electrons system in a quantum dot: J_1 is the vector of relative motion and J_2 is the vector of the center of mass.

$$r_1 = \frac{\mu_e^*}{M} J_1 + J_2, r_2 = -\frac{\mu_e^*}{M} J_1 + J_2, M = 2\mu_e^*, \mu = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} = \frac{1}{2}\mu_e^*, \quad (16)$$

Here μ is the reduced mass of electrons in electron-electron interactions. The Hamiltonians of the center of mass (specified 'c') and the relative motion (specified, 'r') without spin interactions in these variables read

$$H_c = \frac{1}{2}[P_c + A_c]^2 + \frac{4}{\mu_e^*}\hbar^2 V_0\, a_T^2 \rho_c^2 = \frac{P_c^2}{2} + \frac{\hbar^2 \mu_e^*}{4M}\omega_B^2 \rho_c^2 + \frac{\hbar \mu_e^*}{2M}\omega_B L_c + \\ + \frac{4}{\mu_e^*}\hbar^2 V_0\, a_T^2 \rho_c^2 \quad (17)$$

$$H_r = \frac{1}{2}[P_r + A_r]^2 + \frac{1}{4\mu_e^*}\hbar^2 V_0\, a_T^2 \rho_r^2 + \frac{e^2}{8\pi\varepsilon_r\varepsilon_0\rho_r} = \frac{P_r^2}{2\mu} + \frac{\mu_e^*}{16}\omega_B^2 \rho_r^2 + \frac{\hbar}{2} L_r \omega_B + \\ + \frac{1}{4\mu_e^*}\hbar^2 V_0\, a_T^2 \rho_r^2 + \frac{e^2}{8\pi\varepsilon_r\varepsilon_0\rho_r} \quad (18)$$

$$q = \frac{J_1}{\hbar}\sqrt{2\mu_e^*}, \quad \omega_B = \frac{eB}{c\mu_e^*}, \quad \rho_q^2 = q_1^2 + q_2^2, \quad M = 2\mu_e^* \quad , \mu = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} = \frac{1}{2}\mu_e^*$$

$$L = -i\hbar[r.\nabla] \rightarrow \quad L_{c=} - i\hbar\nabla_c . L_r = -i\hbar\nabla_r, J_1 \rightarrow \rho_r, J_2 \rightarrow \rho_c, \quad (19)$$

here operators L is the angular momentum components in the coordinate system of the center of mass (L_c) and relative motion (L_r) along the z axis in the inherent coordinate systems. The conservation of angular momentum and the two centers' adiabatic approximation leads to components that can be used to determine the wave function and eigenenergy of strong electron-electron interactions in QDs. Therefore, the total wave function reads [12, 15]

$$\Psi(J_1.J_2) = \phi(J_1)\Phi(J_2)\chi(S_i, S_j) \rightarrow \Psi(J_1.J_2) = \frac{e^{i\ell\varphi}}{\sqrt{\pi}}\mathcal{R}(J_1.J_2), \quad (20)$$

where ℓ is the orbital quantum number, $S_{i,j}$ is spin, $\phi(J_1)$ is the wave function of internal confined electron-electron interactions in QDs, $\Phi(J_2)$ is the wave function of the center of mass, and $\chi(S_i, S_j)$ is the wave function of spin interactions. Therefore, the Hamiltonian of the center of mass and relative motion based on the non-local spin-spin interaction between electrons quantum dots $H_s = 0$, read

$$\left[\frac{P_c^2}{2}+\frac{\hbar^2\mu_e^*}{4M}\omega_B^2\rho_c^2\ +\frac{\hbar\mu_e^*}{2M}\omega_B L_c+\frac{4\hbar^2}{\mu_e^*}V_0\ a_T^2\rho_c^2\ -E_c\right]\Phi_c=0, \tag{21}$$

$$\left[\frac{P_r^2}{2\mu}+\frac{\mu_e^*}{16}\omega_B^2\rho_r^2+\frac{\hbar}{2}L_r\omega_B+\frac{\hbar^2}{4\mu_e^*}V_0\ a_T^2\rho_r^2\ +\frac{e^2}{8\pi\varepsilon_r\varepsilon_0\rho_r}-E_r\right]\phi_r=0. \tag{22}$$

As we have made aware, it is well-known and important a mathematical skill as being able us to solve equations of the Hamiltonian of the center of mass and relative motion of electron-electron interactions in QDs, that is taken to determine the eigenvalue of the center of mass and the wave function for exponential potential at a finite temperature. The eigenvalue of the center of mass is determined:

$$E_{c(n_r.m_\ell)}\ =\frac{\hbar}{2}\omega_B\left(\left(1+\frac{8V_0a_T^2}{\omega_B^2\mu_e^*}\right)^{\frac{1}{2}}(n_r+|m_\ell|+1)+m_\ell\right), \tag{23}$$

where $n_r=0.1.2.\cdots$, is the radial quantum number and $m_\ell=0.\pm1.\pm2.\cdots$, is the azimuthal quantum number. Then the wave function of the center of mass is defined

$$\Phi_{c(n_r.m_\ell)}=\frac{2\ n_r!\ {a_T}^{0.5m_\ell}}{\lambda_0\sqrt{2\pi}(n_r+|m_\ell|)!}\ (A)^{0.5m_\ell}\mathcal{L}_{(n_r.m_\ell)}(Aa_T^2\ )exp(-Aa_T^2\ ), \tag{24}$$

here $\mathcal{L}_{(n_r.m_\ell)}(Aa_T^2)$ is the Laguerre function and set of below notations are used to define the elements and properties of the Hamiltonian of the strong electron-electron system at the zero azimuthal angles:

$$\lambda=\frac{\lambda_0}{\sqrt{a_T}}=\frac{1}{\sqrt{a_T}}\left(\frac{\hbar^2}{2V_0\mu_e^*}\right)^{\frac{1}{4}},\quad \eta=\left(\frac{8V_0a_T^2}{\mu_e^*}+\omega_B^2\right)^{\frac{1}{2}},\quad A=\frac{8\rho_r^2V_0}{\hbar\eta}, \tag{25}$$

and the Hamiltonian of relative motion of electron-electron interactions in quantum dots at finite temperature from (21) reads

$$\left[\frac{-1}{2}\left[\frac{\partial^2}{\partial\rho_r^2}+\frac{2|m_\ell|+1}{\rho_r}\frac{\partial}{\partial\rho_r}\right]+\frac{\hbar^2\rho_r^2}{4}\left(\frac{2V_0\ a_T^2}{\mu_e^*}+\frac{\omega_B^2}{4}\right)^{\frac{1}{2}}+\frac{e^2}{8\pi\varepsilon_r\varepsilon_0\rho_r}-E_r\right]\phi_r=0, \tag{26}$$

then after some careful calculation and the pure mathematical analysis of this equation, the energy eigenvalue of the relative mass is defined

$$E_{r(n_r.m_\ell)} = \frac{\hbar}{2}\left(\frac{2V_0a_T^2}{\mu_e^*}\right)^{\frac{1}{2}}\left[\frac{m_\ell}{4}\frac{\omega_B^2\mu_e^*}{V_0a_T^2} + Aa_T^2(2n_r + |m_\ell| + 1)\left(1 + \frac{1}{8}\left(\frac{\omega_B^2\mu_e^*}{V_0a_T^2}\right)\right)^{\frac{1}{2}} + \frac{1.5\Gamma(|m_\ell|+0.5)}{\sqrt{2}\Gamma(|m_\ell|+1)}\frac{\mu_e^*\,e^2}{4\pi\varepsilon_r\varepsilon_0\hbar^2}\lambda\sqrt{A}\,a_T\left(1 + \frac{1}{8}\left(\frac{\omega_B^2\mu_e^*}{V_0a_T^2}\right)\right)^{\frac{1}{4}}\right] \quad (27)$$

A. Quanto-relativistic effects

Recent advances in hi-tech semiconductors and nano quantum dot technologies have made it possible to fabricate nanostructures of different sizes, shapes, thermal, optical, and electrical properties at finite temperatures [16] and some of the different properties based on the spin-orbit interaction effects and excited states [16,17] which is important in the nanoelectronics, micro-laser technologies [18], thermoelectric, nano-optoelectronics [19-22]. Quantum dots technology has a wide range of applications including in the progress of modern technology, information processing, energy physics. When fabricating these structures, it is necessary to include quanto-relativistic characteristics, particle's mass correction, relativistic effect on spin-orbital interactions. The most useful analysis method and techniques to include quanto-orbital corrections in the semiconductor quantum dots are based on the field of quantum field theory, symplectic geometry method, and relativistic electrodynamics. The symplectic geometry method arising from ideas of QFT has been presented to calculate the characteristic of strong electron-electron interactions in quantum dot systems. For exponential potential admitting the existence of strong electron-electron interaction's, there is always a transformation of the canonical operators that leads to a Gaussian asymptotic form for the wave function at large distances. However, the asymptotic manner of the wave functions for large distances does not coincide with this manner. Therefore, we have to modify the variables in the original RSE so that the MRSE should have solutions with the Gaussian asymptotic behavior. In the exponential version of a strong electron-electron potential $U(r,T) = V_0\left(1 - e^{-(a_T r)^2}\right)$, MRSE is performed by going over to the $d+1$ dimension spacetime coordinate, where the wave function of the system becomes the oscillator one. The existence of such a variation, the 3D dimensional spacetime coordinate exponential system can transform into the oscillator one in the 4D dimensional spacetime coordinate and it is necessary to represent the canonical variables coordinate and momentum of the Hamiltonian through the creation and annihilation operators this transformation has been found in Ref. [12,15]. Now, by considering renormalization plus the quanto-relativistic behavior, the modified equation yielded the interaction Hamiltonian in quantum field theory formalism,

the scattering matrix, and the corresponding Feynman diagram [4, 11]. The correlation function between the field's associated current and the quantum numbers calculates the electron's masses. It is related to Green's function and the Feynman functional route integral in quantum mechanics [15]. Consequently, the strong electron-electron interaction's at finite temperature characteristics (relativistic effective mass, energy, etc.) are denoted as the correlation function's asymptotic limit. Therefore, in this article the eigenvalue energy and the relativistic effective electron's mass (μ^*) in MF of two-electron interaction's in a quantum dot with zero radial quantum number and zero angular momentum number are calculated and the behaviors of the energy spectrum of the two-electron QD with temperature relation as a function of the magnetic field strength are plotted. In the next section, the quanto-relativistic effect and correction to the ground state eigenvalue of the center of mass and relative motion Eqs. (14-19) are presented and determined.

4. RELATIVISTIC EFFECTIVE MASS OF ELECTRONS

In Eqs. (23, 27) the electron's effective mass (m_e^*)is the mass that it seems to have when responding to forces or the mass that it seems to have when interacting with other identical particles in a thermal distribution under the external field $B(r)$. One of the results from the quanto-relativistic effects on the strong electron-electron interactions in QDs is that the movement and interactions of electrons in a parabolic-exponential potential field can be very different from their motion in a vacuum [7]. The relativistic effective mass is a quantity that is used to describe the strong electron-electron interactions by modeling the behavior of the relativistic effect of mass. For some purposes and some nanomaterials, the effective mass can be considered to be a simple constant of material and the value of effective mass depends on the purpose for which it is used, and can vary depending on several factors like temperature and quanto-relativistic nature of mass in the strong interactions [7] especially in nano quantum dots. For electrons or electron holes in a solid, the effective mass is usually stated in units of the, m_e ($m_e = 0.5MeV$). In these units, it is usually in the range 0.01 to 10, but can also be lower or higher—for example, reaching 1,000 in exotic heavy fermion materials, or anywhere from zero to infinity (depending on definition) in graphene. As it simplifies the more general band theory, the electronic effective mass can be seen as an important basic parameter that influences measurable properties of a solid, including everything from the efficiency of a solar cell to the

speed of an integrated circuit. Therefore, based on Eq. (9) quanto-relativistic effect on the effective mass of electrons in a quantum dot is defined [12, 15]

$$\mu_e^* = \sqrt{m_e^{*2} - 2\mu^2 E'(\mu, T)}. \quad (28)$$

as we mentioned above $E'(\mu, T) = \frac{\partial E(\mu,T)}{\partial \mu}$, and $E(\mu, T)$is the eigenenergy of the total Hamiltonian in the projective unitary representation and the μ parameter represents the boundary system's component mass (for more details see [12]), and then, the relativistic correction to the component's effective mass in the strong electron-electron interactions in Eqs. (23, 27) can be included. Thus the eigenvalues of the center of mass and the relative motion at a finite temperature and relativistic effect on the effective mass can be replaced. In this research, we used the typical quantum dots for GaAs and then the quanto-relativistic to the ground state energy at a finite temperature with the effective mass $m_e^* = 0.035MeV$ and relativistic effective mass μ_e^* for GaAs are plotted in Fig. 2.

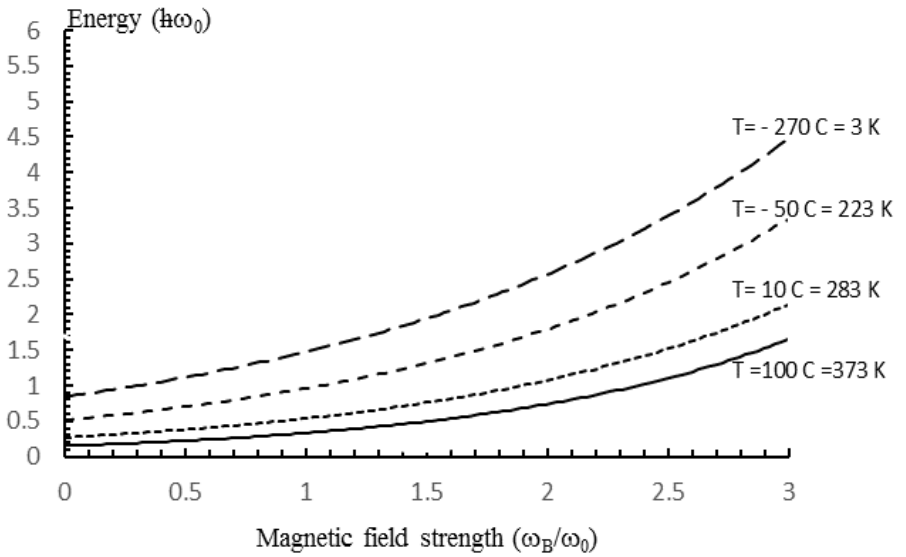

Fig. 2. The energy of the QDs in units of $\hbar\omega_0$, for GaAs is a function of ω_B/ω_0 (magnetic field strength) at various temperature values:
$T = -270C\ (3K), -50C\ (223K), 10C\ (283K), 100C\ (373K)$.

Fig. 2 shows the energy of QDs, at various temperature values. This relativistic correction on energy-temperature behavior is again due to dependence of the physical material parameters ε and m_e^* (effective mass of electrons) which directly depends on relativistic behavior and the temperature value of the quantum dot. Based on theoretical calculation and curves in Fig. 2, the behavior of the energy values of QD as the temperature decreases are observed. The curves are plotted with the quantum numbers $n_r = m_\ell = 0$, and various temperatures. The effective mass of electrons $m_e^* = 0.035MeV$ and in this theoretical article based on the given parameters and constant, the relativistic effective mass has a range $0.045 \leq \mu_e^* \leq 0.075\ eV$, which depends on the pressure, thickness, coupling constant, etc. The results quanto-relativistic corrections to the strong electron-electron interactions quantum dots with the quantum numbers $n_r = m_\ell = 0$, at a finite temperature [23-25] in the magnetic field are shown, the behaviors of the relativistic effects on electron's effective mass. The theoretical results obtained can be compared with values wherever available data and the investigation shows a good agreement with experimental and theoretical results [24]. The combined effects of temperature on the energy levels of a QD under a magnetic field had been studied and described. The exact relativistic effect on mass had been used to solve QD Hamiltonian and to obtain the eigenenergy is the main subject of this article. Within the proposed method the analytical expressions for the energy levels of the QDs are obtained. The CP interaction and the relativistic correction effect are treated exactly and from the analysis of the eigenenergy spectrum, it follows that the action and reaction between the strong electron-electron interaction and MF is an important ingredient for the prediction of the behavior of the ground phase transitions. The results presented here will be useful for the analysis of the electron properties and characteristics in two-electron QDs and will allow us to conclude a deviation of the real confining potential from the relativistic corrections to the mass, and also exact conditions for spin singlet-triplet transitions due to the electron-electron interaction in a QFT and QM can be determined perfectly.

A. Correction to the spin singlet-triplet transition energy

Studies of strong electron-electron interaction states intensified with the construction and formation of quantum systems confined in spatial dimensions known as artificial-exotic atoms, superatoms, or QDs. Recently there has been

great interest in these exotic systems which are the subjects of popular research due to their physical properties for quantum technologies, quantum computers, solar cells, quantum dot lasers, electrically tunable spin qubits, electrical manipulation of individual spins, and transistors, etc. [26-28]. "In 1994 Wagner et al. predicted a transition for the ground state energy from the spin-singlet to the spin-triplet state as a function of the magnetic field [29]." After then, numerical diagonalization of the Hamiltonian matrix, variational approach, Hartree-Fock, and some other methods and techniques are used to determine and calculate the energy levels of QDs. In this article, the PUR method has been presented to determine the eigenenergy levels of QDs based on describing the relationship between temperature and relativistic corrections to the effective mass of two-electron quantum dots in MF. According to the previous paragraphs, we can define and determine spin eigenenergy according to the strong electron-electron interaction in QDs. The benefit of these corrections and correlation effects of the interacting electrons that are confined in a QD can be shown in the theoretical way of the relativistic effect of the spin-orbit coupling that mixes states with different multiplicities, e.g. singlet-triplet spin transition As we know the quantum dots can be a good motive to be a new candidate for future quantum processors and quantum technologies, because of MF effect on the spin transitions and adaptation. Hence, one of the major advantages of the use of the relativistic correction to the mass in the total Hamiltonian is the possibility to determine and consistently yield more accurate results of the spin singlet-triplet transition characteristic in QDs. The total wave function of two electrons QDs is a product of spin and spatial terms. The total parity of partial terms is $P = i^{2m_\ell}$, and has an even parity P^+ for even m_ℓ and P^- for an odd m_ℓ. The total spin parity terms based on the total partial value must be a singlet state $S = 0$ if P^+ (symmetry state) and a triplet $S = 1$ if P^- (antisymmetry state). Hence the total spin interactions can be presented by the relation $S = 0.5|1 - i^{2m_\ell}|$ [29].

We can define the separation form of the total spin Hamiltonian of Eq. (14) in the form: $H_{spin} = H_{LS} + H_{s_{ij}}$, where H_{LS} is the Hamiltonian of the spin-orbit interactions and $H_{s_{ij}}$ is an additional Hamiltonian spin interaction with the eigenenergy named after Zeeman energy which reads $H_{s_{ij}} = g\mu_B(B \cdot \sigma) = g\mu_B(S_i + S_j)$, where the vector of Pauli matrices $\sigma = (\sigma_x + \sigma_y + \sigma_z)$ is the electron spin operator, $S_{i,j}$ is the electron spin [30,31], μ_B is the Bohr magneton, g (g factor) is the phenomenological parameter [32]. The Zeeman splitting of energy levels with different spins in a magnetic field is an important parameter in the growing field of spin-based quantum technologies. Hence, we can define

Zeeman's eigenenergy based on the previous descriptions. The Pauli principle gives us the presentation of the Zeeman's eigenenergy according to the total wave function that must be antisymmetric with respect to canonical transformation leads to the exact spin behavior of the magnetic quantum number corresponding to the relative motion in singlet and triplet states. We do not intend to investigate the effects of spin interactions in this article, so it's worth mentioning that how the relativistic correction of mass and temperature relation can affect the results of the total eigenenergy of a quantum dot [29]. From Eq. (26) and $H_{s_{ij}} = g\mu_B(S_i + S_j)$, one can determine the eigenvalue $E_{s(s,m_\ell)}$ of the Zeeman effect with the relativistic correction on mass reads:

$$E_{s(s,m_\ell)} = \frac{(1-i^{2m_\ell})}{4m_e}\hbar g\mu_e^* \omega_B = \frac{(1-i^{2m_\ell})}{4m_e}\hbar g\omega_B\left(m_e^{*2} - 2\mu^2 E'(\mu,T)\right)^{\frac{1}{2}} . \qquad (29)$$

From Eqs. (23, 27, 29) we can investigate different ground state energy as a function of the relativistic effective mass $\mu_e^* = \left(m_e^{*2} - 2\mu^2 E'(\mu,T)\right)^{\frac{1}{2}}$ of the electron, $\lambda = \frac{1}{\sqrt{a_T}}\left(\frac{\hbar^2}{2V_0\mu_e^*}\right)^{\frac{1}{4}}, a_T, \omega_B, \omega_0 \equiv \left(\frac{2a_T^2 V_0}{\mu_e^*}\right)^{\frac{1}{2}}$. Based on the Eqs. (23, 27, 30) we can realize that the ground state energy of strong electron-electron interaction (the Coulomb potential) two-electron in quantum dot leads to different ground state arrangement $m_\ell = -1,-2,-3, \ldots$, unlike the ground state level in the absence of strong electron-electron interaction (Coulomb potential) $m_\ell = 0$. Therefore, one can define the different arrangement of energy levels in a particular order of singlet and triplet states (the singlet-triplet transition) with the main condition $m_\ell \leq 0$, [29] and we defined the relativistic effective mass correction at finite temperature to the spin singlet-triplet transition energies. Therefore, we showed in Eqs. (29) and (27) that the relativistic effect on mass μ_e^* and finite temperature how to affect the transition energy and the total eigenenergy of the quantum dots in the different states of m_ℓ against ω_B MF frequency for strong electron-electron interacting in the ground state and other higher states.

5. CONCLUSION

The Schrödinger equation's strong electron-electron interaction at a finite temperature in the external magnetic field solution for GaAs QDs with a parabolic-exponential and Coulomb potentials is determined in this paper utilizing the projective unitary representation. In detail, the theory of quantum dot Hamiltonian including the relativistic correction on mass and temperature dependence of the effective mass of GaAs QD material is described. These temperature-dependent and mass parameters should be included in the energy spectrum Eq. (23). For quantum dots made of GaAs the dependency of temperature and relativistic mass are given in Eq. (27). We characterized the temperature dependency of strong electron-electron interaction with the quanto-relativistic corrections to the mass and energy. In strong electron-electron interactions, the relativistic mass-temperature connection was developed. At zero and finite temperatures, we have characterized the electron's effective mass. Temperature dependency was found by treating nonzero temperature as an exponential function and modifying parameters like the Debye mass. The findings establish that strong electron-electron interaction in the two-electron QDs with quanto-relativistic corrections and temperature effect may represent a novel property of hidden characteristics of semiconductors. According to the findings, the study's theoretical results are expected to open new avenues for new theoretical knowledge of nanotechnology due to the work's outstanding and similar results to previous theoretical or experimental studies. Perspectives of quantum field theory and quantum mechanics in SQDs have tremendous scope in nanoelectronics, thermoelectric, and nano-optoelectronics which may revolutionize technology. However, a correct understanding of the energy of quantum dots is necessary for it, especially the quantum confinement energy. This article suggests the relativistic effect on the effective mass of electrons in quantum dots which is determined and defined to be greater than what is available in the standard kinds of literature and articles. Hence, we suggest that careful experiments on quantum dots need to confirm and reveal it. The growing interest in the optoelectrical properties and characteristics of QDs is causing great research interest and excitement in the Hitech generation in the fields of photonics, microelectronics, and optoelectronics. Hence, the theoretical data gained may be used in a future study, potentially opening up new possibilities for identifying unique characteristics of electron-electron and electron-hole systems.

ACKNOWLEDGMENT

This work was supported by scientific programs of the department of physics and engineering sciences, Buein Zahra Higher Education Center of Engineering and Technology under the grant 1999 for developing research.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this manuscript

REFERENCES

[1] A. Singha, V. Pellegrini, A. Pinczuk, et al., *Strong electron-electron interactions in Si/SiGe quantum dots*, Phys. Rev. Lett. 104, 246802, 2010. https://doi.org/10.1103/PhysRevLett.104.246802

[2] D. Miserev and O. Sushkov, *Prediction of the spin-triplet two-electron quantum dots in Si: Towards controlled quantum simulations of magnetic systems*, Phys. Rev. B, 100(20), 5129, 2019, https://link.aps.org/doi/10.1103/PhysRevB.100.205129; R. Gould, *Quantum electrodynamics. Electromagnetic Processes*, 1st ed. Springer- Verlag, 2020, https://doi.org/10.1515/9780691215846-008.

[3] F. Caruso, V. Oguri, and F. Silvera, *How the inter-electronic potential Ansätze affect the bound state solutions of a planar two-electron quantum dot model*, Physica E: Low-dimensional Systems, 105, 182, 2019. https://doi.org/10.1016/j.physe.2018.09.017

[4] K. Heyong-chan, et al., *New variational perturbation theory based on−deformed oscillator*, International Journal of Theoretical Physics, 45, 1017, 2006. https://doi.org/10.1007/s10773-006-9094-3.

[5] A. Taqi, J. Diouri, *A theoretical model for exciton binding energies in rectangular and parabolic spherical finite quantum dots*, Semiconductor physics quantum electronics & optoelectronics, (15)4, 365, 2012. https://doi.org/10.15407/spqeo15.04.365

[6] M. Hartstein, Y. Hsu, K. A. Modic, et al., *Hard antinodal gap revealed by quantum oscillations in the pseudogap regime of underdoped high-Tc superconductors*, Nature Physics, 4,2020. https://doi.org/10.1038/s41567-

020-0910-0

[7] M. El-Said, *Two-electron quantum dots in a magnetic field*, Semiconductor science, and technology. 10(10),1310, 1995. https://doi.org/10.1088/0268-1242/10/10/003

[8] E. Tazikeh, et al., *Optoelectronical Properties of a Metalloid-Doped B12N12 Nano-Cage*, Journal of Optoelectronical Nanostructures, 5(1), 10, 2020. https://doi.org/20.1001.1.24237361.2020.5.1.7.2

[9] M., Rushka, J. Freericks, *A completely algebraic solution of the simple harmonic oscillator*, Am J Phys. 88(11), 976, 2019. https://doi.org/10.1119/10.0001702

[10] W. Greiner, S. Schramm, E. Stein, *Quantum chromodynamics*, Springer Science & Business Media; 2007. https://link.springer.com/content/pdf/bfm%3A978-3-540-48535-3%2F1.pdf

[11] D. Fujiwara, *Rigorous Time Slicing Approach to Feynman Path Integrals*, 1st ed. Springer International Publishing, 2017. https://doi.org/10.1007/978-4-431-56553-6

[12] M. Dienykhan, G. Efimov, G. Ganbold, N. Nedelko, *Oscillator representation in quantum physics (lecture notes in physics monographs)*, 1st ed. Springer International Publishing, 1995, https://doi.org/10.1007/978-3-540-49186-6; M. Martin Nieto, *Existence of bound states in continuous $0<D<\infty$ dimensions,* Physics Letters A, 293(1-2), 10, 2020. https://doi.org/10.1016/S0375-9601(01)00827-1

[13] J. Kelley, J. Leventhal, *Ladder operators for the harmonic oscillator. Problems in Classical and Quantum Mechanics*, Springer International Publishing, 2017. https://doi.org/10.1007/978-3-319-46664-4

[14] A. Maireche, *A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential*, Sri Lankan J Phys, 21(1), 11, 2020. https://doi.org/10.4038/sljp.v21i1.8069

[15] A. Jahanshir, *Relativistic modification of the exciton's mass in monolayer TMDCs materials*, Journal of Advanced Materials and Processing, 8(4), 2020. https://doi.org/20.1001.1.2322388.2020.8.4.5.0

[16] S. J. Mousavi, *First–Principle Calculation of the Electronic and Optical Properties of Nanolayered ZnO Polymorphs by PBE and mBJ Density*

Functionals, Journal of Optoelectronical Nanostructures, 2(4), 1, 2017. https://doi.org/20.1001.1.24237361. 2017.2.4.1.1.; Ghaffary, et al., *Study of the spin-orbit interaction effects on energy levels and the absorption coefficients of spherical quantum dot and quantum anti-dot under the magnetic field*, Journal of Optoelectronical Nanostructures, 6(2), 55, 2021. https://doi.org/10.30495/jopn.2021.27965.1222

[17] M. Ciurla, J. Adamowski, B. Szafran, S. Bednare, *Modelling of confinement potentials in quantum dots*, Physica E, 15, 261 (2002). https://doi.org/10.1016/S1386-9477(02)00572-6.

[18] M. Mohebbifar, *Study of the Quantum Efficiency of Semiconductor Quantum Dot Pulsed Micro-Laser*, Journal of Optoelectronical Nanostructures, 6(1), 59, 2021. https://doi.org/10.30495/jopn.2021.4544.

[19] M. Dharma-wardana, et al., *Correlation functions in electron-electron and electron-hole double quantum wells: temperature, density and barrier-width dependence*, arXiv:1901.00895v1 [cond.mat. mes-hall] 2019, https://doi.org/110.1103/PhysRevB.99.035303; A. Tanveer Karim, et al., *Temperature dependency of excitonic effective mass and charge carrier conduction mechanism in CH3NH3PbI3−xClx thin films*, Scientific Reports, 11,10772, 2021. https://doi.org/10.1038/s41598-021-90247-x

[20] O. Ciftja, *Understanding electronic systems in semiconductor quantum dots*, Physica Scri. 88(5), 058302, 2013. https://doi.org/10.1088/0031-8949/88/05/058302

[21] A. Jahanshir, A. Tarasenko, *Computational method for determining the bound state oscillator*, Math. Comp. Sci. Journal, 1(3), 16-22, 2021, https://doi.org/10.30511/mcs.2021.526310.1018.

[22] A. Abu Alia, et al., *Effect of tilted electric field and magnetic field on the energy levels, binding energies and heat capacity of a donor impurity in GaAs quantum dot*, Indian Journal of Pure & Applied Physics 59, 365, 2021. http://nopr.niscair.res.in/handle/123456789/57471

[23] A. Abu Alia, K. M. Elsaid and A. Shaer, *Magnetic properties of GaAs parabolic quantum dot in the presence of donor impurity under the influence of external tilted electric and magnetic fields*, Journal of Taibah University

for Science, 13(1), 687, 2019. https://doi.org/10.1080/16583655.2019.1622242

[24] F. Bozour, M. Elsaid, K, Ilaiwi, *The effects of pressure and temperature on the energy levels of a parabolic two-electron quantum dot in a magnetic field*, Journal of King Saud University, 33(7), 2017. https://doi.org/10.1016/j.jksus.2017.01.001

[25] S.Chaudhuri, *Two-electron quantum dot in a magnetic field: Analytic solution for a finite potential model*, Physica E: Low-dimensional Systems and Nanostructures, 128, 114571, 2021. https://doi.org/10.1016/j.physe.2020.114571.

[26] D. Jirovec, *Dynamics of hole singlet-triplet qubits with large g-factor differences*, arXiv:2111.05130v1 [cond-mat. mes-hall], 2021. https://arxiv.org/abs/2111.05130v1

[27] G. Scappucci, et al., *The germanium quantum information route*, Nat. Rev. Mater, 6, 926-943, 2021. https://doi.org/10.1038/s41578-020-00262-z

[28] D. Loss, D. P. DiVincenzo, *Quantum computation with quantum dots*, Phys. Rev. A57, 120–126, 1998. https://doi.org/10.1103/PhysRevA.57.120.

[29] M. Wagner, U. Merkt, A.V. Chaplik, *Spin-singlet–spin-triplet oscillations in quantum dots*, Phy. Rev.B45, 1951, 1992. https://doi.org/10.1103/PhysRevB.45.1951

[30] L.D. Landau, E.M. Lifschitz, *Quantum Mechanics nonrelativistic Theory*, 3st ed Pergamon Oxford, 1977.

http://power1.pc.uec.ac.jp/~toru/notes/LandauLifshitz-QuantumMechanics.pdf

[31] S. J. Prado, S. J. Prado, *Zeeman effect and magnetic field induced spin-hybridization in semiconductor quantum dots*, Journal of Physics-Condensed Matter 16(39), 6949, 2004. https://doi.org/10.1088/0953-8984/16/39/027.

[32] P. Stano, *g-factor of electrons in gate-defined quantum dots in a strong in-plane magnetic field*, Phys. Rev. B98, 195314, 2018. https://doi.org/10.1103/PhysRevB.98.195314.